Closed geodesics on positively curved Finsler spheres

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Abstract

In this paper, we prove that for every Finsler n-sphere $(S^n,\, F)$ for $n \geq 3$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exists one elliptic closed geodesic whose linearized Poincaré map has at least one eigenvalue which is of the form $\exp(\pi i\mu)$ with an irrational μ. Furthermore, there always exist three prime closed geodesics on any $(S^3,\, F)$ satisfying the above pinching condition.

Key words: Finsler spheres, closed geodesics, index iteration, mean index identity, stability.

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1 Introduction and main results

This paper is devoted to a study on closed geodesics on Finsler n-spheres. For the definition of closed geodesics on a Finsler manifold, we refer readers to [BCS1] and [She1]. As usual, on any Finsler n-sphere $S^n = (S^n, F)$ a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to S^n$ is *prime*, if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. Note that on a non-symmetric Finsler sphere, c^{-1} is not a geodesic in general. We call two prime closed geodesics c and d *distinct*, if there is no $\theta \in S^1$ such that $c(t) = d(t+\theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* for short when we talk about more than one prime closed geodesics. On a symmetric

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Finsler (or Riemannian) n-sphere, two closed geodesics c and d are called *geometrically distinct*, if $c(S^1) \neq d(S^1)$, i.e., their image sets in S^n are distinct.

For a closed geodesic c on (S^n, F), denote by P_c the linearized Poincaré map of c. Then $P_c \in \mathrm{Sp}(2n-2)$ is symplectic. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} | \, |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$. A closed geodesic c is called *elliptic* if $e(P_c) = 2(n-1)$, i.e., all the eigenvalues of P_c locate on $\mathbf{U}$. Following H-B. Rademacher in [Rad3], the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold (M, F) is defied to be

$$\lambda := \max\{F(-X) \,|\, X \in TM, \, F(X) = 1\} \geq 1.$$

It was quite surprising when Katok [Kat1] in 1973 found some non-symmetric Finsler metrics on S^n with only finitely many prime closed geodesics and all closed geodesics are non-degenerate and elliptic. The smallest number of closed geodesics that one obtains in these examples is $2n$ on S^{2n} and S^{2n-1} (cf. [Zil1]). Then it is an open question whether there are always at least n prime closed geodesics on any Finsler n-sphere (cf. p.156 of [Zil1]).

The following are the main results in this paper:

Theorem 1.1. *For every Finsler n-sphere (S^n, F) for $n \geq 3$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exists one elliptic closed geodesic whose linearized Poincaré map has at least one eigenvalue which is of the form $\exp(\pi i \mu)$ with an irrational μ. The number μ could be called a Floquet exponent.*

Theorem 1.2. *Under the assumption of Theorem 1.1 and suppose the number of prime closed geodesics is finite. Denote by $\{P_{c_j}\}_{1 \leq j \leq p}$ the linearized Poincaré maps of them. Then there exist at least two pairs of eigenvalues of those P_{c_j}s which are of the form $\exp(\pi i \mu)$ with μ being irrational. (the two pairs of eigenvalues maybe belong to one P_{c_j}).*

Remark 1.3. Note that on the standard Riemannian n-sphere of constant curvature 1, all geodesics are closed and their linearized Poincaré maps are I_{2n-2}, i.e., the identity matrix in $\mathbf{R}^{2n-2}$ and then there exists no eigenvalue of them which is an irrational multiple of π. Hence our above theorems describe a character of a Finsler sphere which carries finitely many prime closed geodesics.

Note also that our definition of ellipticity is different from that in [BTZ1], in which they call a closed geodesic c is of *elliptic-parabolic type* if the linearized Poincaré map P_c of c splits into two-dimensional rotations and a part whose eigenvalues are ± 1.

In [BTZ1], W. Ballmann, G. Thorbergsson and W. Ziller proved that if a Riemannian manifold $M = S^n$ satisfies $\frac{9}{16} \le K \le 1$, there exists a prime closed geodesic of elliptic-parabolic type on M. In [Rad4], H-B. Rademacher proved that if a Finsler manifold $M = S^n$ satisfies $\frac{9}{4}\frac{\lambda^2}{(\lambda+1)^2} < K \le 1$ with $\lambda < 2$, there exists a prime closed geodesic of elliptic-parabolic type on M. Comparing with their results, we can prove the following

Theorem 1.4. *Under the assumption of Theorem 1.1 and suppose the number of prime closed geodesics is finite. Then there exists a closed geodesic c (not necessarily to be prime) of elliptic-parabolic type in the sense of [BTZ1].*

Note that by Theorem 7 in [Rad4], the existence of at least two prime closed geodesics on any Finsler 3-sphere (S^3, F) satisfying the assumption of Theorem 1.1. Now we can prove the following

Theorem 1.5. *For every Finsler 3-sphere (S^3, F) with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \le 1$, there exist at least three prime closed geodesics.*

Remark 1.6. Note that our Theorem 1.5 gives a partial result to the problem (4) in p.156 in [Zil1] for the S^3 case. As mentioned above, there are examples with exactly four prime closed geodesics on the 3-sphere. In [BTZ2], W. Ballmann, G. Thorbergsson and W. Ziller proved that for a Riemannian metric on S^n with sectional curvature $1/4 \le K \le 1$ there exist $g(n)$ geometrically distinct closed geodesics, and $g(3) = 4$. In [LoW1], Y. Long and the author proved that for every Riemannian 3-sphere (M, g) with injectivity radius $\mathrm{inj}(M) \ge \pi$ and the sectional curvature K satisfying $\frac{1}{16} < K \le 1$ there exist at least two geometrically distinct closed geodesics. However, the proofs of these results relies essentially on the symmetry of the metric, hence it can not be used here. The method in [LoW1] works only in the study of the existence of two closed geodesics, while that in the present paper works for more that two closed geodesics.

Our proof of these theorems contains mainly three ingredients: the common index jump theorem of Y. Long, Morse theory, and an existence theorem of N. Hingston. Fix a Finsler metric F on S^n. Let $\Lambda = \Lambda S^n$ be the free loop space of S^n, which is a Hilbert manifold. For definition and basic properties of Λ, we refer readers to [Kli2] and [Kli3]. Let $E(c) = \frac{1}{2}\int_0^1 F(\dot{c}(t))^2 dt$ be the energy functional on Λ. In this paper for $\kappa \in \mathbf{R}$ we denote by

$$\Lambda^\kappa = \{d \in \Lambda \,|\, E(d) \le \kappa\}, \tag{1.1}$$

and consider the quotient space Λ/S^1. Since the energy functional E is S^1-invariant, the negative gradient flow of E induce a flow on Λ/S^1, so we can apply Morse theory on Λ/S^1. By a result of H-B. Rademacher in [Rad1] of 1989, we get the Morse series of the space pair $(\Lambda/S^1, \Lambda^0/S^1)$ with

rational coefficients. The reason we use $(\Lambda/S^1, \Lambda^0/S^1)$ instead of (Λ, Λ^0) is that the Morse series of the first is lacunary.

Sections 2 to 4 are preliminary materials for our proof. In Section 2, basic properties of critical modules of closed geodesics are introduced, whose proofs can be found in [Rad2] and [BaL1]. In Section 3, Morse inequalities on the quotient space $(\Lambda/S^1, \Lambda^0/S^1)$ are given, whose proof can be found in [Rad1] and [Rad2]. In Section 4, by results in [Lon1] of 2000 of Y. Long, we give the classification of closed geodesics on S^n. N. Hingston's Theorem in [Hin1] is also listed in Section 4.

In Section 5, we establish a mean index equality when there exist only finitely many prime closed geodesics on (S^n, F). An abstract version of such an equality was established by H-B. Rademacher in [Rad2]. Since we need this equality with exact coefficients. we give a complete proof for it in Section 5.

Based on these preparations, our Theorems 1.1-1.4 are proved in Section 6. As an application of these theorems, Theorem 1.5 is proved in Section 7.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. Let (p, q) denotes the greatest common devisor of p and $q \in \mathbf{N}$. We use only singular homology modules with $\mathbf{Q}$-coefficients. For terminologies in algebraic topology we refer to [GrH1]. For $k \in \mathbf{N}$, we denote by $\mathbf{Q}^k$ the direct sum $\mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$ of k copies of $\mathbf{Q}$ and $\mathbf{Q}^0 = 0$. For an S^1-space X, we denote by $\overline{X}$ the quotient space X/S^1. We define the functions

$$\begin{cases} [a] = \max\{k \in \mathbf{Z} \,|\, k \le a\}, \quad \mathcal{E}(a) = \min\{k \in \mathbf{Z} \,|\, k \ge a\}, \\ \qquad \varphi(a) = \mathcal{E}(a) - [a], \end{cases} \tag{1.2}$$

Especially, $\varphi(a) = 0$ if $a \in \mathbf{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbf{Z}$.

2 Critical modules of iterations of closed geodesics

In this section, we will study critical modules of closed geodesic, all the details can be found in [Rad2] or [BaL1].

On a compact Finsler manifold (M, F), we choose an auxiliary Riemannian metric. This endows the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ with a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries, cf. [Kli2], Chapters 1 and 2. This action is defined by translating the parameter, i.e.

$$(s \cdot \gamma)(t) = \gamma(t + s)$$

for all $\gamma \in \Lambda$ and $s, t \in S^1$. The Finsler metric F defines an energy functional E and a length functional L on Λ by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\dot{\gamma}(t))^2 dt, \quad L(\gamma) = \int_{S^1} F(\dot{\gamma}(t)) dt. \tag{2.1}$$

Both functionals are invariant under the S^1-action. The critical points of E of positive energies are precisely the closed geodesics $c : S^1 \to M$ of the Finsler structure. If $c \in \Lambda$ is a closed geodesic then c is a regular curve, i.e. $\dot{c}(t) \neq 0$ for all $t \in S^1$, and this implies that the second differential $E''(c)$ of E at c exists. As usual we define the index $i(c)$ of c as the maximal dimension of subspaces of $T_c\Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of c so that $\nu(c) + 1$ is the dimension of the null space of $E''(c)$.

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi^m : \Lambda \to \Lambda$ by

$$\phi^m(\gamma)(t) = \gamma(mt) \qquad \forall\, \gamma \in \Lambda, t \in S^1. \tag{2.2}$$

We also use the notation $\phi^m(\gamma) = \gamma^m$. For a closed geodesic c, the mean index is defined to be:

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}. \tag{2.3}$$

If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$ then γ is called *prime*. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde{\gamma} \in \Lambda$ such that $\gamma = \tilde{\gamma}^m$.

For a closed geodesic c we set

$$\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}.$$

If $A \subseteq \Lambda$ is invariant under some subgroup Γ of S^1, we denote by A/Γ the quotient space of A with respect to the action of Γ.

Using singular homology with rational coefficients we will consider the following critical $\mathbf{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_*(E, c) = H_*\left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1\right). \tag{2.4}$$

In order to relate the critical modules to the index and nullity of c we use the results of D. Gromoll and W. Meyer from [GrM1], [GrM2]. Following [Rad2], Section 6.2, we introduce finite-dimensional approximations to Λ. We choose an arbitrary energy value $a > 0$ and $k \in \mathbf{N}$ such that every F-geodesic of length $< \sqrt{2a/k}$ is minimal. Then

$$\Lambda(k, a) = \left\{\gamma \in \Lambda \mid E(\gamma) < a \text{ and } \gamma|_{[i/k,(i+1)/k]} \text{ is an } F\text{-geodesic for } i = 0, \ldots, k-1\right\}$$

is a $(k\cdot\dim M)$-dimensional submanifold of Λ consisting of closed geodesic polygons with k vertices. The set $\Lambda(k,a)$ is invariant under the subgroup $\mathbf{Z}_k$ of S^1. Closed geodesics in $\Lambda^{a-} = \{\gamma \in \Lambda \mid E(\gamma) < a\}$ are precisely the critical points of $E|_{\Lambda(k,a)}$, and for every closed geodesic $c \in \Lambda(k,a)$ the index of $(E|_{\Lambda(k,a)})''(c)$ equals $i(c)$ and the null space of $(E|_{\Lambda(k,a)})''(c)$ coincides with the nullspace of $E''(c)$, cf. [Rad2], p.51.

We call a closed geodesic satisfying the isolation condition, if the following holds:

(Iso) For all $m \in \mathbf{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of E.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

Now we can apply the results by D. Gromoll and W. Meyer [GrM1] to a given closed geodesic c satisfying (Iso). If $m = m(c)$ is the multiplicity of c, we choose a finite-dimensinal approximation $\Lambda(k,a) \subseteq \Lambda$ containing c such that m divides k. Then the isotropy subgroup $\mathbf{Z}_m \subseteq S^1$ of c acts on $\Lambda(k,a)$ by isometries. Let D be a $\mathbf{Z}_m$-invariant local hypersurface transverse to $S^1 \cdot c$ in $c \in D$. According to [GrM1], Lemma 1, for every such D we can find a product neighborhood $B_+ \times B_- \times B_0$ of $0 \in \mathbf{R}^{\dim\Lambda(k,a)-1}$ such that $\dim B_- = i(c), \dim B_0 = \nu(c)$, and a diffeomorphism

$$\psi : B = B_+ \times B_- \times B_0 \to \psi(B_+ \times B_- \times B_0) \subseteq D$$

from B onto an open subset $\psi(B) \subseteq D$ such that $\psi(0) = c$ and ψ is $\mathbf{Z}_m$-invariant, and there exists a smooth function $f : B_0 \to \mathbf{R}$ satisfying

$$f'(0) = 0 \quad \text{and} \quad f''(0) = 0 \tag{2.5}$$

and

$$E \circ \psi(x_+, x_-, x_0) = |x_+|^2 - |x_-|^2 + f(x_0), \tag{2.6}$$

for $(x_+, x_-, x_0) \in B_+ \times B_- \times B_0$. As usual, we call

$$N = \{\psi(0,0,x_0) | x_0 \in B_0\}$$

a local characteristic manifold at c and

$$U = \{\psi(0,x_-,0) | x_- \in B_-\}$$

a local negative disk at c, N and U are $\mathbf{Z}_m$-invariant. It follows from (2.6) that c is an isolated critical point of $E|_N$. We set $N^- = N \cap \Lambda(c)$, $U^- = U \cap \Lambda(c) = U \setminus \{c\}$ and $D^- = D \cap \Lambda(c)$. Using (2.6), the fact that c is an isolated critical point of $E|_N$, and the Künneth formula, one concludes

$$H_*(D^- \cup \{c\}, D^-) = H_*(U^- \cup \{c\}, U^-) \otimes H_*(N^- \cup \{c\}, N^-), \tag{2.7}$$

where

$$H_q(U^- \cup \{c\}, U^-) = H_q(U, U \setminus \{c\}) = \begin{cases} \mathbf{Q}, & \text{if } q = i(c), \\ 0, & \text{otherwise}, \end{cases} \tag{2.8}$$

cf. [Rad2], Lemma 6.4 and its proof. As studied in p.59 of [Rad2], for all $m \in \mathbf{N}$, let respectively

$$H_*(X, A)^{\pm \mathbf{Z}_m} = \{[\xi] \in H_*(X, A) \,|\, T_*[\xi] = \pm[\xi]\}, \tag{2.9}$$

where T is a generator of the $\mathbf{Z}_m$-action.

The following Propositions were proved in [Rad2] and [BaL1].

Proposition 2.1. (cf. Satz 6.11 of [Rad2] or Proposition 3.12 of [BaL1]) *Let c be a prime closed geodesic on a Finsler manifold (M, F) satisfying (Iso). Then we have*

$$\begin{aligned} \overline{C}_q(E, c^m) \quad &\equiv H_q\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right) \\ &= \left(H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})\right)^{+\mathbf{Z}_m} \end{aligned}$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbf{Q}, & \text{if } \; i(c^m) - i(c) \in 2\mathbf{Z}, \text{ and } q = i(c^m) \\ 0, & \text{otherwise}. \end{cases} \tag{2.10}$$

(ii) When $\nu(c^m) > 0$, there holds

$$\overline{C}_q(E, c^m) = H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\beta(c^m)\mathbf{Z}_m}, \tag{2.11}$$

where $\beta(c^m) = 1$ (or $= -1$), when $i(c^m) - i(c)$ is even (or odd).

In order to study the degenerate part of the local critical modules, we need the following result which follows from Satz 6.6 of [Rad2] directly.

Lemma 2.2. *Let c be a prime closed geodesic on a Finsler manifold (M, F) satisfying (Iso). Then there holds*

$$H_q(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{+\mathbf{Z}_m} = H_q(N^-_{c^m} \cup \{c^m\}/\mathbf{Z}_m, N^-_{c^m}/\mathbf{Z}_m), \qquad \forall\, q \in \mathbf{Z},\, m \in \mathbf{N}.$$

We introduce the following

Definition 2.3. *Suppose c is a closed geodesic of multiplicity $m(c) = m$ satisfying (Iso). If N is a local characteristic manifold at c, $N^- = N \cap \Lambda(c)$ and $j \in \mathbf{Z}$, we define*

$$k_j(c) \;\; \equiv \;\; \dim H_j(N^- \cup \{c\}, N^-), \tag{2.12}$$

$$k_j^{\pm 1}(c) \;\; \equiv \;\; \dim H_j(N^- \cup \{c\}, N^-)^{\pm \mathbf{Z}_m}. \tag{2.13}$$

Clearly the integers $k_j(c)$ and $k_j^{\pm 1}(c)$ equal to 0 when $j < 0$ or $j > \nu(c)$ and can take only values 0 or 1 when $j = 0$ or $j = \nu(c)$.

Lemma 2.4. (cf. Lemma 2.4 of [LoW1]) *Let c be a prime closed geodesic on a Finsler manifold (M,F) satisfying (Iso).*

(i) There holds

$$0 \le k_j^{\pm 1}(c^m) \le k_j(c^m), \qquad \forall\, m \in \mathbf{N},\ j \in \mathbf{Z}. \tag{2.14}$$

(ii) For any $m \in \mathbf{N}$, there holds

$$k_0^{+1}(c^m) = k_0(c^m), \quad k_0^{-1}(c^m) = 0. \tag{2.15}$$

(iii) In particular, if c^m is non-degenerate, i.e. $\nu(c^m) = 0$, then

$$k_0^{+1}(c^m) = k_0(c^m) = 1, \quad k_0^{-1}(c^m) = 0. \tag{2.16}$$

Following the ideas of Gromoll-Meyer on the degenerate part of the critical module in Theorem 3 of [GrM2], we have the following result.

Proposition 2.5. (cf. Theorem 3 of [GrM2], Section 7.1 of [Rad2] and Theorem 3.11 of [BaL1]) *Let c be a prime closed geodesic on a Finsler manifold (M,F) satisfying (Iso). Suppose for some integer $m = np \ge 2$ with n and $p \in \mathbf{N}$ the nullitities satisfy*

$$\nu(c^m) = \nu(c^n).$$

Then the following holds for the degenerate part of the critical module of E with coefficient $\mathbf{Q}$.

(i) For any integer j, there hold

$$\begin{aligned} &H_j(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-) = H_j(N_{c^n}^- \cup \{c^n\}, N_{c^n}^-), \\ &k_j(c^m) = k_j(c^n). \end{aligned} \tag{2.17}$$

(ii) For any integer j, there hold

$$\begin{aligned} H_j(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{\pm \mathbf{Z}_m} &= H_j(N_{c^n}^- \cup \{c^n\}, N_{c^n}^-)^{\pm \mathbf{Z}_n}, \\ k_j^{\pm 1}(c^m) &= k_j^{\pm 1}(c^n). \end{aligned} \tag{2.18}$$

Proposition 2.6. (cf. Satz 6.13 of [Rad2]) *Let c be a prime closed geodesic on a Finsler manifold (M,F) satisfying (Iso). For any $m \in \mathbf{N}$, we have*

(i) If $k_0(c^m) = 1$, there holds $k_j^{\pm 1}(c^m) = 0$ for $1 \le j \le \nu(c^m)$.

(ii) If $k^{+1}_{\nu(c^m)}(c^m) = 1$ or $k^{-1}_{\nu(c^m)}(c^m) = 1$, there holds $k^{\pm 1}_j(c^m) = 0$ for $0 \le j \le \nu(c^m) - 1$.

(iii) If $k^{+1}_j(c^m) \ge 1$ or $k^{-1}_j(c^m) \ge 1$ for some $1 \le j \le \nu(c^m) - 1$, there holds $k^{\pm 1}_{\nu(c^m)}(c^m) = 0 = k_0(c^m)$.

(iv) In particular, if $\nu(c^m) \le 2$, then only one of the $k_j(c^m)$'s can be non-zero.

Proposition 2.7. *Let c be a prime closed geodesic on a Finsler manifold (M, F) satisfying (Iso). Suppose for some integer $m = np \ge 2$ with n and $p \in \mathbf{N}$ the nullities satisfy*

$$\nu(c^m) \ge \nu(c^n).$$

Suppose further $k_{\nu(c^m)}(c^m) = 1$, then we have $k_{\nu(c^n)}(c^n) = 1$. In particular, we have

$$k_j(c^n) = 0, \quad 0 \le j \le \nu(c^n) - 1. \tag{2.19}$$

Proof. The proof follows directly from Lemma 5 of [GrM1] as well as Section 7.1 of [Rad2], we now describe it briefly. Let D_{c^m} be a $\mathbf{Z}_m$-invariant local hypersurface transverse to $S^1 \cdot c^m$ in $c^m \in D_{c^m}$ as above and similarly for D_{c^n}. Then the p-fold iteration map ϕ^p maps D_{c^n} into D_{c^m}. Since $E|D_{c^m}$ is $\mathbf{Z}_p$-invariant, $\mathrm{grad}E(d^p)$ is targential to the fixed point set $\phi^p(D_{c^n})$ for $d \in D_{c^n}$ and this yields

$$\mathrm{grad}E(d^p) = \phi^p_*(\mathrm{grad}E(d)), \quad \forall d \in D_{c^n}. \tag{2.20}$$

Now the proof of Lemma 1 in [GrM1] yields that ϕ^p embeds N_{c^n} into N_{c^m} as a submanifold, where N_{c^n} and N_{c^m} are local characteristic manifolds at c^n and c^m respectively. Note that

$$E(\phi^p(d)) = p^2 E(d), \quad \forall d \in N_{c^n}. \tag{2.21}$$

By Corollary 8.4 of [MaW1], $k_{\nu(c^m)}(c^m) = 1$ if and only if c^m is a local maximum of E in N_{c^m}. This together with (2.21) imply that c^n is a local maximum of E in N_{c^n} too. Hence we use Corollary 8.4 of [MaW1] again to obtain the proposition. ∎

3 The structure of $H_*(\overline{\Lambda} S^n, \overline{\Lambda}^0 S^n; \mathbf{Q})$

In this section, we briefly describe the relative homological structure of the quotient space $\overline{\Lambda} \equiv \overline{\Lambda} S^n$. Here we have $\overline{\Lambda}^0 = \overline{\Lambda}^0 S^n = \{\text{constant point curves in } S^n\} \cong S^n$.

Theorem 3.1. (H.-B. Rademacher, Theorem 2.4 and Remark 2.5 of [Rad1]) *We have the Poincaré series*

(i) When $n = 2k+1$ is odd

$$\begin{aligned} P(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n)(t) &= t^{n-1}\left(\frac{1}{1-t^2} + \frac{t^{n-1}}{1-t^{n-1}}\right) \\ &= t^{2k}\left(\frac{1}{1-t^2} + \frac{t^{2k}}{1-t^{2k}}\right). \end{aligned} \tag{3.1}$$

Thus for $q \in \mathbf{Z}$ and $l \in \mathbf{N}_0$, we have

$$\begin{aligned} b_q &= b_q(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n) \\ &= \mathrm{rank} H_q(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n) \\ &= \begin{cases} 2, & \textit{if} \quad q \in \{4k+2l, \quad l = 0 \bmod k\}, \\ 1, & \textit{if} \quad q \in \{2k\} \cup \{2k+2l, \quad l \neq 0 \bmod k\}, \\ 0 & \textit{otherwise}. \end{cases} \end{aligned} \tag{3.2}$$

(ii) When $n = 2k$ is even

$$\begin{aligned} P(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n)(t) &= t^{n-1}\left(\frac{1}{1-t^2} + \frac{t^{n(m+1)-2}}{1-t^{n(m+1)-2}}\right)\frac{1-t^{nm}}{1-t^n} \\ &= t^{2k-1}\left(\frac{1}{1-t^2} + \frac{t^{4k-2}}{1-t^{4k-2}}\right), \end{aligned} \tag{3.3}$$

where $m = 1$ by Theorem 2.4 of [Rad1]. Thus for $q \in \mathbf{Z}$ and $l \in \mathbf{N}_0$, we have

$$\begin{aligned} b_q &= b_q(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n) \\ &= \mathrm{rank} H_q(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n) \\ &= \begin{cases} 2, & \textit{if} \quad q \in \{6k-3+2l, \quad l = 0 \bmod 2k-1\}, \\ 1, & \textit{if} \quad q \in \{2k-1\} \cup \{2k-1+2l, \quad l \neq 0 \bmod 2k-1\}, \\ 0 & \textit{otherwise}. \end{cases} \end{aligned} \tag{3.4}$$

We have the following version of the Morse inequality.

Theorem 3.2. (Theorem 6.1 of [Rad2]) *Suppose that there exist only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ on (M, F), and $0 \le a < b \le \infty$ are regular values of the energy functional E. Define for each $q \in \mathbf{Z}$,*

$$\begin{aligned} M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) &= \sum_{1\le j\le p,\ a<E(c_j^m)<b} \mathrm{rank}\overline{C}_q(E, c_j^m) \\ b_q(\overline{\Lambda}^b, \overline{\Lambda}^a) &= \mathrm{rank} H_q(\overline{\Lambda}^b, \overline{\Lambda}^a). \end{aligned}$$

Then there holds

$$\begin{aligned} M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) &- M_{q-1}(\overline{\Lambda}^b, \overline{\Lambda}^a) + \cdots + (-1)^q M_0(\overline{\Lambda}^b, \overline{\Lambda}^a) \\ &\ge b_q(\overline{\Lambda}^b, \overline{\Lambda}^a) - b_{q-1}(\overline{\Lambda}^b, \overline{\Lambda}^a) + \cdots + (-1)^q b_0(\overline{\Lambda}^b, \overline{\Lambda}^a), \end{aligned} \tag{3.5}$$

$$M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) \ge b_q(\overline{\Lambda}^b, \overline{\Lambda}^a). \tag{3.6}$$

4 Classification of closed geodesics on S^n and existence theorem

Let c be a closed geodesic on a Finsler n-sphere $S^n = (S^n,\, F)$. Denote the linearized Poincaré map of c by $P_c \in \mathrm{Sp}(2n-2)$. Then P_c is a symplectic matrix. Note that the index iteration formulae in [Lon1] of 2000 (cf. Chap. 8 of [Lon2]) work for Morse indices of iterated closed geodesics (cf. [LLo1], Chap. 12 of [Lon2]). Since every closed geodesic on a sphere must be orientable. Then by Theorem 1.1 of [Liu1] of C. Liu (cf. also [Wil1]), the initial Morse index of a closed geodesic c on a n-dimensional Finsler sphere coincides with the index of a corresponding symplectic path introduced by C. Conley, E. Zehnder, and Y. Long in 1984-1990 (cf. [Lon2]).

As in §1.8 of [Lon2], define the homotopy component $\Omega^0(P_c)$ of P_c to be the path component of $\Omega(P_c)$, where

$$\begin{aligned}\Omega(P_c) = \{N \in Sp(2n-2) \mid \quad & \sigma(N)\cap U = \sigma(P_c)\cap U,\ and \\ & \nu_\lambda(N) = \nu_\lambda(P_c)\ \forall \lambda \in \sigma(P_c)\cap U\}.\end{aligned} \tag{4.1}$$

The next theorem is due to Y. Long (cf. Theorem 8.3.1 and Corollary 8.3.2 of [Lon2]).

Theorem 4.1. *Let $\gamma \in \{\xi \in C([0,\tau], Sp(2n)) \mid \xi(0) = I\}$, Then there exists a path $f \in C([0,1], \Omega^0(\gamma(\tau))$ such that $f(0) = \gamma(\tau)$ and*

$$\begin{aligned}f(1) = \quad & N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \diamond N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1,u_1) \diamond \cdots \diamond N_2(\omega_{r_*},u_{r_*}) \\ & \diamond N_2(\lambda_1,v_1) \diamond \cdots \diamond N_2(\lambda_{r_0},v_{r_0}) \diamond M_0\end{aligned} \tag{4.2}$$

where $N_2(\omega_j,u_j)$s are non-trivial and $N_2(\lambda_j,v_j)$s are trivial basic normal forms; $\sigma(M_0)\cap U = \emptyset$; p_-, p_0, p_+, q_-, q_0, q_+, r, r_* and r_0 are non-negative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; θ_j, α_j, $\beta_j \in (0,\pi)\cup(\pi,2\pi)$; these integers and real numbers are uniquely determined by $\gamma(\tau)$. Then using the functions defined in (1.2).

$$\begin{aligned}i(\gamma,m) = \quad & m(i(\gamma,1)+p_-+p_0-r) + 2\sum_{j=1}^{r}\mathcal{E}\left(\frac{m\theta_j}{2\pi}\right) - r - p_- - p_0 \\ & -\frac{1+(-1)^m}{2}(q_0+q_+) + 2\left(\sum_{j=1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right) - r_*\right).\end{aligned} \tag{4.3}$$

$$\begin{aligned}\nu(\gamma,m) = \quad & \nu(\gamma,1) + \frac{1+(-1)^m}{2}(q_-+2q_0+q_+) + 2(r+r_*+r_0) \\ & -2\left(\sum_{j=1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right) + \sum_{j=1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right) + \sum_{j=1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right)\right)\end{aligned} \tag{4.4}$$

$$\hat{i}(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi}. \tag{4.5}$$

Where $N_1(1, \pm 1) = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, $N_1(-1, \pm 1) = \begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}$, $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ *with some* $\theta \in (0, \pi) \cup (\pi, 2\pi)$ *and* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2\times 2}$*, such that* $(b_2 - b_3)\sin\theta > 0$*, if* $N_2(\omega, b)$ *is trivial;* $(b_2 - b_3)\sin\theta < 0$*, if* $N_2(\omega, b)$ *is non-trivial. We have* $i(\gamma, 1)$ *is odd if* $f(1) = N_1(1, 1)$*,* I_2*,* $N_1(-1, 1)$*,* $-I_2$*,* $N_1(-1, -1)$ *and* $R(\theta)$*;* $i(\gamma, 1)$ *is even if* $f(1) = N_1(1, -1)$ *and* $N_2(\omega, b)$*;* $i(\gamma, 1)$ *can be any integer if* $\sigma(f(1)) \cap \mathbf{U} = \emptyset$*.*

We will use the following theorem of N. Hingston in the S^n case below. Note that the proof of N. Hingston's theorem does not use the special properties of Riemannian metric, hence it holds for Finsler metric as well.

Theorem 4.2. (Follows from Proposition 1 of [Hin1], cf. Lemma 3.4.12 of [Kli3]) *Let* c *be a closed geodesic of length* L *on a Finsler* n*-sphere* $S^n = (S^n, F)$ *such that as a critical orbit of the energy functional* E *on* ΛS^n*, every orbit* $S^1 \cdot c^m$ *of its iteration* c^m *is isolated. Suppose*

$$i(c^m) + \nu(c^m) \le m(i(c) + \nu(c)) - (n-1)(m-1), \qquad \forall m \in \mathbf{N}, \tag{4.6}$$

$$k_{\nu(c)}(c) \ne 0. \tag{4.7}$$

Then S^n *has infinitely many prime closed geodesics.*

Note that in (4.7), we have used the Shifting theorem in [GrM1]. Especially, (4.7) means that c is a local maximum in the local characteristic manifold N_c at c.

5 A mean index equality on (S^n, F)

In this section, suppose that there are only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ on (S^n, F) with $\hat{i}(c_j) > 0$ for $1 \le j \le p$. We establish an equality of c_js involving their mean indices.

Definition 5.1. *Let* $M = (M, F)$ *be a compact Finsler manifold of dimension* n*,* c *be a prime closed geodesic on* M *satisfying (Iso). For each* $m \in \mathbf{N}$*, the critical type numbers of* c^m *is defined by the following* $2n-1$ *tuple of integers via Definition 2.3*

$$\begin{aligned} K(c^m) &\equiv (k_0^{\beta}(c^m), k_1^{\beta}(c^m), \ldots, k_{2n-2}^{\beta}(c^m)) \\ &= (k_0^{\beta(c^m)}(c^m), k_1^{\beta(c^m)}(c^m), \ldots, k_{\nu(c^m)}^{\beta(c^m)}(c^m), 0, \ldots, 0), \end{aligned} \tag{5.1}$$

where $\beta = \beta(c^m) = (-1)^{i(c^m) - i(c)}$*. We call a prime closed geodesic* c *homologically invisible if* $K(c^m) = 0$ *for all* $m \in \mathbf{N}$ *and homologically visible otherwise.*

As Lemma 2 of [GrM2], we have

Lemma 5.2. *Let c be a prime closed geodesic on a compact Finsler manifold (M,F) satisfying (Iso). Then there exists a minimal $T(c)\in\mathbf{N}$ such that*

$$\nu(c^{p+T(c)})=\nu(c^p),\quad i(c^{p+T(c)})-i(c^p)\in 2\mathbf{Z},\qquad \forall p\in\mathbf{N}, \tag{5.2}$$

$$K(c^{p+T(c)})=K(c^p),\qquad \forall p\in\mathbf{N}. \tag{5.3}$$

We call $T(c)$ the minimal period of critical modules of iterations of c.

Proof. Denote the linearized Poincaré map of c by $P_c:\mathbf{R}^{2(n-1)}\to\mathbf{R}^{2(n-1)}$. Then P_c is a symplectic matrix. Denote by $\lambda_i=exp(\pm 2\pi\frac{r_i}{s_i})$ the eigenvalues of P_c possessing rotation angles which are rational multiple of π with $r_i,\ s_i\in\mathbf{N}$ and $(r_i,s_i)=1$ for $1\le i\le q$. Let $T(c)$ be the least common multiple of $s_1,\ldots,s_q$. Then the first equality in (5.2) holds. If the second equality in (5.2) does not hold, replace $T(c)$ by $2T(c)$. Then the later conclusion in (5.2) follows from Theorem 9.3.4 of [Lon2].

In order to prove (5.3), it suffices to show

$$K(c^{m+qT(c)})=K(c^m),\qquad \forall q\in\mathbf{N},\ 1\le m\le T(c). \tag{5.4}$$

In fact, assume that (5.4) is proved. Note that (5.3) follows from (5.4) with $q=1$ directly when $p\le T(c)$. When $p>T(c)$, we write $p=m+qT(c)$ for some $q\in\mathbf{N}$ and $1\le m\le T(c)$. Then by (5.4) we obtain

$$K(c^{p+T(c)})=K(c^{m+(q+1)T(c)})=K(c^m)=K(c^{m+qT(c)})=K(c^p),$$

i.e., (5.3) holds.

To prove (5.4), we fix an integer $m\in[1,T(c)]$. Let

$$A=\{s_i\in\{s_1,\ldots,s_q\}\mid\ s_i \text{ is a factor of } m\},$$

and let m_1 be the least common multiple of elements in A. Hence we have $m=m_1m_2$ for some $m_2\in\mathbf{N}$ and $\nu(c^m)=\nu(c^{m_1})$. Thus by Proposition 2.5 we have $k_l^{\beta(c^m)}(c^m)=k_l^{\beta(c^m)}(c^{m_1})$. Since $m+pT(c)=m_1m_3$ for some $m_3\in\mathbf{N}$, we have by Proposition 2.5 that $k_l^{\beta(c^{m+pT(c)})}(c^{m+pT(c)})=k_l^{\beta(c^{m+pT(c)})}(c^{m_1})$. By (5.2), we obtain $\beta(c^{m+pT(c)})=\beta(c^m)$, and then (5.4) is proved. This completes the proof. ∎

Definition 5.3. *The Euler characteristic $\chi(c^m)$ of c^m is defined by*

$$\begin{aligned}\chi(c^m)\ &\equiv\ \chi\left((\Lambda(c^m)\cup S^1\cdot c^m)/S^1,\Lambda(c^m)/S^1\right),\\ &\equiv\ \sum_{q=0}^{\infty}(-1)^q\dim\overline{C}_q(E,c^m).\end{aligned} \tag{5.5}$$

Here $\chi(A,B)$ denotes the usual Euler characteristic of the space pair (A,B).

The average Euler characteristic $\hat{\chi}(c)$ of c is defined by

$$\hat{\chi}(c) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\le m\le N} \chi(c^m). \tag{5.6}$$

The following remark shows that $\hat{\chi}(c)$ is well-defined and is a rational number.

Remark 5.4. By (5.5), we have

$$\chi(c^m) = \sum_{q=0}^{\infty} (-1)^q \dim \overline{C}_q(E, c^m) = \sum_{l=0}^{2n-2} (-1)^{i(c^m)+l} k_l^{\beta(c^m)}(c^m). \tag{5.7}$$

Here the second equality follows from Proposition 2.1 and Definition 5.1. Hence by (5.6) and Lemma 5.2 we have

$$\begin{aligned} \hat{\chi}(c) &= \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\le m\le N \\ 0\le l\le 2n-2}} (-1)^{i(c^m)+l} k_l^{\beta(c^m)}(c^m) \\ &= \lim_{s\to\infty} \frac{1}{sT(c)} \sum_{\substack{1\le m\le T(c),\ 0\le l\le 2n-2 \\ 0\le p<s}} (-1)^{i(c^{pT(c)+m})+l} k_l^{\beta(c^{pT(c)+m})}(c^{pT(c)+m}) \\ &= \frac{1}{T(c)} \sum_{\substack{1\le m\le T(c) \\ 0\le l\le 2n-2}} (-1)^{i(c^m)+l} k_l^{\beta(c^m)}(c^m) = \frac{1}{T(c)} \sum_{1\le m\le T(c)} \chi(c^m). \end{aligned} \tag{5.8}$$

Therefore $\hat{\chi}(y)$ is well defined and is a rational number.

Let (X,Y) be a space pair such that the Betti numbers $b_i = b_i(X,Y) = \dim H_i(X,Y;\mathbf{Q})$ are finite for all $i \in \mathbf{Z}$. As usual the *Poincaré series* of (X,Y) is defined by the formal power series $P(X,Y) = \sum_{i=0}^{\infty} b_i t^i$. Suppose there exist only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ and satisfy $\hat{i}(c_j) > 0$ for $1 \le j \le p$ on (S^n, F). The *Morse series* $M(t)$ of the energy functional E of the space pair $(\Lambda S^n/S^1, \Lambda^0 S^n/S^1)$ is defined as usual by

$$M(t) = \sum_{\substack{q\ge 0,\ m\ge 1 \\ 1\le j\le p}} \dim \overline{C}_q(E, c_j^m) t^q.$$

Then Theorem 3.2 yields a formal power series $Q(t) = \sum_{i=0}^{\infty} q_i t^i$ with nonnegative integer coefficients q_i such that

$$M(t) = P(\Lambda S^n/S^1, \Lambda^0 S^n/S^1)(t) + (1+t)Q(t). \tag{5.9}$$

For a formal power series $R(t) = \sum_{i=0}^{\infty} r_i t^i$, we denote by $R^n(t) = \sum_{i=0}^{n} r_i t^i$ for $n \in \mathbf{N}$ the corresponding truncated polynomials. Using this notation, (5.9) becomes

$$(-1)^I q_I = M^I(-1) - P^I(-1), \qquad \forall I \in \mathbf{N}. \tag{5.10}$$

By Satz 7.8 of [Rad2] we have

$$\begin{aligned}
& \lim_{I\to\infty}\frac{1}{I}P^I(\Lambda S^n/S^1,\Lambda^0 S^n/S^1)(-1) \\
= \ & B(n,m) \\
= \ & \begin{cases} \frac{-m(m+1)n}{2n(m+1)-4}, & n \quad even, \\ \frac{n+1}{2(n-1)} & n \quad odd, \end{cases}
\end{aligned} \tag{5.11}$$

where $m=1$ by Corollary 2.6 of [Rad1].

The following consequence of an important result by H.-B. Rademacher (Theorem 7.9 in [Rad2]) is needed in Sections 6 below. Here we have derived precise dependence of coefficients on prime closed geodesics in the mean index equality in Theorem 7.9 of [Rad2]. This precise dependence is also crucial for our proofs in Sections 6 below.

Theorem 5.5. *Suppose that there exist only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ with $\hat{i}(c_j)>0$ for $1\le j\le p$ on (S^n,F). Then the following identity holds*

$$\sum_{1\le j\le p}\frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(n,1). \tag{5.12}$$

Proof. Because $\dim\overline{C}_q(E,c_j^m)$ can be non-zero only for $q=i(c_j^m)+l$ with $0\le l\le 2n-2$ by Proposition 2.1, the formal Poincaré series $M(t)$ becomes

$$M(t) = \sum_{\substack{1\le j\le p,\ 0\le l\le 2n-2 \\ m\ge 1}} k_l^{\beta}(c_j^m)t^{i(c_j^m)+l} = \sum_{\substack{1\le j\le p,\ 0\le l\le 2n-2 \\ 1\le m\le T_j,\ s\ge 0}} k_l^{\beta}(c_j^m)t^{i(c_j^{sT_j+m})+l}, \tag{5.13}$$

where we denote by $T_j=T(c_j)$ for $1\le j\le p$. The last equality follows from Lemma 5.2. Write $M(t)=\sum_{h=0}^{\infty}w_h t^h$. Then we have

$$w_h = \sum_{\substack{1\le j\le p,\ 0\le l\le 2n-2 \\ 1\le m\le T_j}} k_l^{\beta}(c_j^m)\,{}^{\#}\{s\in\mathbf{N}_0 \,|\, i(c_j^{sT_j+m})+l=h\}. \tag{5.14}$$

Claim 1. $\{w_h\}_{h\ge 0}$ *is bounded.*

In fact, we have

$$\begin{aligned}
{}^{\#}\{s\in\mathbf{N}_0 \quad | \quad & i(c_j^{sT_j+m})+l=h\} \\
= \quad & {}^{\#}\{s\in\mathbf{N}_0 \,|\, i(c_j^{sT_j+m})+l=h,\ |i(c_j^{sT_j+m})-(sT_j+m)\hat{i}(c_j)|\le n-1\} \\
\le \quad & {}^{\#}\{s\in\mathbf{N}_0 \,|\, |h-l-(sT_j+m)\hat{i}(c_j)|\le n-1\} \\
= \quad & {}^{\#}\left\{s\in\mathbf{N}_0 \,\middle|\, \frac{h-l-(n-1)-m\hat{i}(c_j)}{T_j\hat{i}(c_j)}\le s\le\frac{h-l+(n-1)-m\hat{i}(c_j)}{T_j\hat{i}(c_j)}\right\} \\
\le \quad & \frac{2(n-1)}{T_j\hat{i}(c_j)}+1,
\end{aligned}$$

where the first equality follows from the fact $|i(c^m) - m\hat{i}(c)| \le n-1$ (cf. Theorem 1.4 of [Rad1]). Hence Claim 1 holds.

We estimate next $M^I(-1)$. By (5.8) we obtain

$$\begin{aligned} M^I(-1) &= \sum_{h=0}^{I} w_h(-1)^h \\ &= \sum_{\substack{1\le j\le p,\ 0\le l\le 2n-2\\ 1\le m\le T_j}} (-1)^{i(c_j^m)+l} k_l^{\beta}(c_j^m)\, {}^{\#}\{s\in \mathbf{N}_0 \mid i(c_j^{sT_j+m})+l\le I\}. \end{aligned} \tag{5.15}$$

Here the latter equality holds by Lemma 5.2.

Claim 2. *There is a real constant $C>0$ independent of I, but depend on c_j for $1\le j\le p$ such that*

$$\left| M^I(-1) - \sum_{\substack{1\le j\le p,\ 0\le l\le 2n-2\\ 1\le m\le T_j}} (-1)^{i(c_j^m)+l} k_l^{\beta}(c_j^m) \frac{I}{T_j\hat{i}(c_j)} \right| \le C, \tag{5.16}$$

where the sum in the left hand side of (5.16) equals to $I\sum_{1\le j\le p}\frac{\hat{\chi}(c_j)}{\hat{i}(c_j)}$ by (5.8).

In fact, we have the estimates

$$\begin{aligned} {}^{\#}\{s\in \mathbf{N}_0 \quad | \quad & i(c_j^{sT_j+m})+l\le I\} \\ &= {}^{\#}\{s\in \mathbf{N}_0 \mid i(c_j^{sT_j+m})+l\le I,\ |i(c_j^{sT_j+m}) - (sT_j+m)\hat{i}(c_j)| \le n-1\} \\ &\le {}^{\#}\{s\in \mathbf{N}_0 \mid 0\le (sT_j+m)\hat{i}(c_j) \le I-l+(n-1)\} \\ &= {}^{\#}\left\{ s\in \mathbf{N}_0 \;\middle|\; 0\le s\le \frac{I-l+(n-1)-m\hat{i}(c_j)}{T_j\hat{i}(c_j)} \right\} \\ &\le \frac{I-l+(n-1)}{T_j\hat{i}(c_j)}+1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} {}^{\#}\{s\in \mathbf{N}_0 \quad | \quad & i(c_j^{sT_j+m})+l\le I\} \\ &= {}^{\#}\{s\in \mathbf{N}_0 \mid i(c_j^{sT_j+m})+l\le I,\ |i(c_j^{sT_j+m}) - (sT_j+m)\hat{i}(c_j)| \le n-1\} \\ &\ge {}^{\#}\{s\in \mathbf{N}_0 \mid i(c_j^{sT_j+m}) \le (sT_j+m)\hat{i}(c_j)+n-1\le I-l\} \\ &\ge {}^{\#}\left\{ s\in \mathbf{N}_0 \;\middle|\; 0\le s\le \frac{I-l-(n-1)-m\hat{i}(c_j)}{T_j\hat{i}(c_j)} \right\} \\ &\ge \frac{I-l-(n-1)}{T_j\hat{i}(c_j)}-2, \end{aligned}$$

where $m\le T_j$ is used. Combining these two estimates together with (5.15), we obtain (5.16).

By Claim 1, the sequence $\{w_h\}_{h\ge 0}$ is bounded. Hence by (5.9), the coefficient sequence $\{q_h\}_{h\ge 0}$ of $Q(t)$ is bounded. Dividing both sides of (5.10) by I, and letting I tend to infinity, together with

Claim 2 and (5.11) we obtain

$$\lim_{I\to\infty}\frac{1}{I}M^I(-1)=\lim_{I\to\infty}\frac{1}{I}P^I(-1)=B(n,1).$$

Hence (5.12) holds by (5.16). ∎

6 Stability of closed geodesics on $(S^n,\ F)$

In this section, we give the proofs of Theorems 1.1, 1.2 and 1.4 by using the mean index identity in Theorem 5.5, Morse inequality and the index iteration theory developed by Y. Long and his coworkers.

In the following for the notation introduced in Section 3 we use specially $M_j = M_j(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n)$ and $b_j = b_j(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n)$ for $j = 0, 1, 2, \ldots$.

Proof of Theorem 1.1. First note that if the flag curvature K of (S^n, F) satisfies $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \le 1$, then every nonconstant closed geodesic must satisfy

$$i(c)\ge n-1. \tag{6.1}$$

This follows from Theorem 3 and Lemma 3 of [Rad3]. Now it follows from Theorem 2.2 of [LoZ1] (Theorem 10.2.3 of [Lon2]) that

$$i(c^{m+1})-i(c^m)-\nu(c^m)\ge i(c)-\frac{e(P_c)}{2}\ge 0,\quad \forall m\in\mathbf{N}. \tag{6.2}$$

Here the last inequality holds by (6.1) and the fact that $e(P_c)\le 2(n-1)$.

Next we prove the theorem by showing that: If the number of prime closed geodesics is finite, then there must exist at least one elliptic closed geodesic whose linearized Poincaré map has at least one eigenvalue which is an irrational multiple of π.

In the rest of this paper, we will assume the following

(F) There are only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ on $(S^n,\ F)$.

The proof contains two steps. In the first step, we prove the existence of at least one elliptic closed geodesic. In the second step, we use Theorem 4.2 and the ideas of step 1 to complete the proof of the theorem.

Step 1. *Claim: Under the assumption (F), there must exist at least one elliptic closed geodesic.*

We prove it by contradiction, i.e., suppose that $e(P_{c_j}) < 2n-2$ for $1\le j\le p$, where P_{c_j} denotes the linearized Poincaré map of c_j. Since $e(P_{c_j})$ is always even, we have

$$e(P_{c_j})\le 2n-4,\quad 1\le j\le p. \tag{6.3}$$

Note that by (6.1) and (4.5), we have $\hat{i}(c_j) > 0$ for $1 \le j \le p$. Actually, we have $\hat{i}(c_j) > n-1$ for $1 \le j \le p$ under the pinching assumption by Lemma 2 of [Rad4]. Hence we can use the common index jump theorem (Theorem 4.3 of [LoZ1], Theorem 11.2.1 of [Lon2]) to obtain some $(N, m_1, \ldots, m_p) \in \mathbf{N}^{p+1}$ such that

$$\begin{aligned} i(c_j^{2m_j}) &\ge 2N - \frac{e(P_{c_j})}{2}, &(6.4)\\ i(c_j^{2m_j}) + \nu(c_j^{2m_j}) &\le 2N + \frac{e(P_{c_j})}{2}, &(6.5)\\ i(c_j^{2m_j+1}) &= 2N + i(c_j). &(6.6) \end{aligned}$$

Moreover $\frac{m_j\theta}{\pi} \in \mathbf{Z}$, whenever $e^{\sqrt{-1}\theta} \in \sigma(P_{c_j})$ and $\frac{\theta}{\pi} \in \mathbf{Q}$. More precisely, by Theorem 4.1 of [LoZ1] (in (11.1.10) in Theorem 11.1.1 of [Lon2], with $D_j = \hat{i}(c_j)$, we have

$$m_j = \left(\left[\frac{N}{M\hat{i}(c_j)}\right] + \xi_j\right) M, \quad 1 \le j \le p, \tag{6.7}$$

where $\xi_j = 0$ or 1 for $1 \le j \le p$ and $\frac{M\theta}{\pi} \in \mathbf{Z}$, whenever $e^{\sqrt{-1}\theta} \in \sigma(P_{c_j})$ and $\frac{\theta}{\pi} \in \mathbf{Q}$ for some $1 \le j \le p$. Furthermore, by (11.1.20) in Theorem 11.1.1 of [Lon2], for any $\epsilon > 0$, we can choose N and $\{\xi_j\}_{1\le j\le p}$ such that

$$\left|\frac{N}{M\hat{i}(c_j)} - \left[\frac{N}{M\hat{i}(c_j)}\right] - \xi_j\right| < \epsilon < \frac{1}{1 + \sum_{1\le j\le p} 4M|\hat{\chi}(c_j)|}, \quad 1 \le j \le p. \tag{6.8}$$

Now by (6.1)-(6.6), we have

$$\begin{aligned} i(c_j^m) + \nu(c_j^m) &\le i(c_j^{2m_j}), \quad \forall m < 2m_j, &(6.9)\\ i(c_j^{2m_j}) + \nu(c_j^{2m_j}) &\le 2N + n - 2, &(6.10)\\ i(c_j^m) &\ge 2N + n - 1, \quad \forall m > 2m_j. &(6.11) \end{aligned}$$

By Theorem 5.5 and (5.11), we have

$$\sum_{1\le j\le p} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(n,1) \in \mathbf{Q}. \tag{6.12}$$

Note by the proof of Theorem 4.1 of [LoZ1] (Theorem 11.1.1 of [Lon2]), we can require that $N \in \mathbf{N}$ further satisfies (cf. (11.1.22) in [Lon2])

$$2NB(n,1) \in \mathbf{Z}. \tag{6.13}$$

Multiplying both sides of (6.12) by $2N$ yields

$$\sum_{1\le j\le p} \frac{2N\hat{\chi}(c_j)}{\hat{i}(c_j)} = 2NB(n,1). \tag{6.14}$$

Claim 1. *We have*

$$\sum_{1\le j\le p} 2m_j\hat{\chi}(c_j) = 2NB(n,1). \tag{6.15}$$

In fact, by (6.12), we have

$$\begin{aligned}
& 2NB(n,1) \\
= \quad & \sum_{1\le j\le p} \frac{2N\hat{\chi}(c_j)}{\hat{i}(c_j)} \\
= \quad & \sum_{1\le j\le p} 2\hat{\chi}(c_j)\left(\left[\frac{N}{M\hat{i}(c_j)}\right]+\xi_j\right)M + \sum_{1\le j\le p} 2\hat{\chi}(c_j)\left(\frac{N}{M\hat{i}(c_j)} - \left[\frac{N}{M\hat{i}(c_j)}\right] - \xi_j\right)M \\
= \quad & \sum_{1\le j\le p} 2m_j\hat{\chi}(c_j) + \sum_{1\le j\le p} 2M\hat{\chi}(c_j)\epsilon_j. \qquad (6.16)
\end{aligned}$$

By Lemma 5.2 and our choice of M, we have

$$\frac{2m_j}{T(c_j)} \in \mathbf{N}, \quad 1\le j\le p. \tag{6.17}$$

Hence (5.8) implies that

$$2m_j\hat{\chi}(c_j) \in \mathbf{Z}, \quad 1\le j\le p. \tag{6.18}$$

Now Claim 1 follows by (6.8), (6.16), (6.13) and (6.18).

Claim 2. *We have*

$$\sum_{1\le j\le p} 2m_j\hat{\chi}(c_j) = M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2}M_{2N+n-2}. \tag{6.19}$$

In fact, by definition, the right hand side of (6.19) is

$$RHS = \sum_{\substack{q\le 2N+n-2 \\ m\ge 1,\ 1\le j\le p}} (-1)^q \dim \overline{C}_q(E, c_j^m). \tag{6.20}$$

By (6.9)-(6.11) and Proposition 2.1, we have

$$\begin{aligned}
RHS \quad &= \sum_{\substack{1\le j\le p,\ 1\le m\le 2m_j \\ q\le 2N+n-2}} (-1)^q \dim \overline{C}_q(E, c_j^m), && (6.21)\\
&= \sum_{1\le j\le p,\ 1\le m\le 2m_j} \chi(c_j^m), && (6.22)
\end{aligned}$$

where the second equality follows from (5.7), (6.9)-(6.10) and Proposition 2.1.

By Lemma 5.2, (5.7)-(5.8) and (6.17), we have

$$\begin{aligned}
\sum_{1\le m\le 2m_j} \chi(c_j^m) \quad &= \quad \sum_{\substack{0\le s<2m_j/T(c_j) \\ 1\le m\le T(c_j)}} \chi(c_j^{sT(c_j)+m}) \\
&= \quad \frac{2m_j}{T(c_j)} \sum_{1\le m\le T(c_j)} \chi(c_j^m) \\
&= \quad 2m_j\hat{\chi}(c_j), && (6.23)
\end{aligned}$$

This proves Claim 2.

In order to complete Step 1, we have to consider the following two cases according to the parity of n.

Case 1. $n = 2k+1$ is odd.

In this case, we have by (5.11)

$$B(n,1) = \frac{n+1}{2(n-1)} = \frac{k+1}{2k}. \tag{6.24}$$

By the proof of Theorem 4.1 of [LoZ1] (Theorem 11.1.1 of [Lon2]), we may further assume $N = mk$ for some $m \in \mathbf{N}$.

Thus by (6.15), (6.19) and (6.24), we have

$$M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2} M_{2N+n-2} = m(k+1). \tag{6.25}$$

On the other hand, we have by (3.2)

$$\begin{aligned}
& b_0 - b_1 + b_2 - \cdots + (-1)^{2N+n-2} b_{2N+n-2} \\
= \quad & b_{2k} + (b_{2k+2} + \cdots + b_{4k} + \cdots + b_{2mk+2} + \cdots + b_{2mk+2k}) - b_{2mk+2k} \\
= \quad & 1 + m(k-1+2) - 2 \\
= \quad & m(k+1) - 1.
\end{aligned} \tag{6.26}$$

In fact, we cut off the sequence $\{b_{2k+2}, \ldots, b_{2mk+2k}\}$ into m pieces, each of them contains k terms. Moreover, each piece contain 1 for $k-1$ times and 2 for one time. Thus (6.26) holds.

Now by Theorem 3.2 and (6.26), we have

$$\begin{aligned}
-m(k+1) \quad &= \quad M_{2N+n-2} - M_{2N+n-3} + \cdots + M_1 - M_0 \\
&\geq \quad b_{2N+n-2} - b_{2N+n-3} + \cdots + b_1 - b_0 \\
&= \quad -(m(k+1) - 1).
\end{aligned} \tag{6.27}$$

This contradiction yields Step 1 for n being odd.

Case 2. $n = 2k$ is even.

In this case, we have by (5.11)

$$B(n,1) = \frac{-2n}{4n-4} = \frac{-k}{2k-1}. \tag{6.28}$$

As in Case 1, we may assume $N = m(2k-1)$ for some $m \in \mathbf{N}$.

Thus by (6.15), (6.19) and (6.28), we have

$$M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2} M_{2N+n-2} = -2mk. \tag{6.29}$$

On the other hand, we have by (3.4)

$$\begin{aligned}
& b_0 - b_1 + b_2 - \cdots + (-1)^{2N+n-2} b_{2N+n-2} \\
= \quad & -b_{2k-1} - (b_{2k+1} + \cdots + b_{6k-3} + \cdots + b_{(m-1)(4k-2)+2k+1} + \cdots + b_{m(4k-2)+2k-1}) \\
& + b_{m(4k-2)+2k-1} \\
= \quad & -1 - m(2k - 2 + 2) + 2 \\
= \quad & -2mk + 1.
\end{aligned} \tag{6.30}$$

In fact, we cut off the sequence $\{b_{2k+1}, \ldots, b_{m(4k-2)+2k-1}\}$ into m pieces, each of them contains $2k-1$ terms. Moreover, each piece contain 1 for $2k-2$ times and 2 for one time. Thus (6.30) holds.

Now by (6.29)-(6.30) and Theorem 3.2, we have

$$\begin{aligned}
-2mk &= M_{2N+n-2} - M_{2N+n-3} + \cdots + M_1 - M_0 \\
&\geq b_{2N+n-2} - b_{2N+n-3} + \cdots + b_1 - b_0 \\
&= -2mk + 1.
\end{aligned} \tag{6.31}$$

This contradiction yields Step 1 for n being even.

Step 2. Under the assumption (F), there must exist one elliptic closed geodesic whose linearized Poincaré map has at least one eigenvalue which is of the form $\exp(\pi i \mu)$ with an irrational μ.

In fact, we shall prove a more stronger result. Denote by $\{P_{c_j}\}_{1\leq j\leq p}$ the linearized Poincaré maps of them. Suppose $\{M_{c_j}\}_{1\leq j\leq p}$ are the basic normal form decompositions of $\{P_{c_j}\}_{1\leq j\leq p}$ in $\{\Omega^0(P_{c_j})\}_{1\leq j\leq p}$ as in Theorem 4.1. Then we have the following

Claim 3. *There must exist $d \in \{c_j\}_{1\leq j\leq p}$ such that the following hold:*

(i) $e(P_d) = 2n-2$, i.e., d is elliptic.

(ii) M_d does not contain $N_1(1,1)$s, $N_1(-1,-1)$s and nontrivial $N_2(\omega, b)$s.

(iii) Any trivial $N_2(\omega, b)$ contained in M_d must satisfies $\frac{\theta}{\pi} \in \mathbf{Q}$, where $\omega = e^{\sqrt{-1}}\theta$.

(iv) If M_d contains $R(\theta)$ for some $\frac{\theta}{\pi} \notin \mathbf{Q}$, then M_d does not contain $R(2\pi - \theta)$.

(v) $k_{\nu(d^{T(d)})}^{\beta(d^{T(d)})}(d^{T(d)}) \neq 0$. Hence $d^{T(d)}$ is a local maximum of the energy functional in the local characteristic manifold at $d^{T(d)}$.

(vi) M_d must contain a term $R(\theta)$ with $\frac{\theta}{\pi} \notin \mathbf{Q}$.

In fact, we first show that there must exist d satisfying (i)-(iv). Suppose none of the closed geodesics in $\{c_j\}_{1\le j\le p}$ satisfies all of (i)-(iv). Then as in Step 1, we obtain some $(N,m_1,\ldots,m_p)\in \mathbf{N}^{p+1}$ such that (6.4)-(6.6) hold.

By Step 1, we have found an elliptic closed geodesic c for which (6.3) does not hold anymore. Our following argument is to find other conditions to replace (6.3), then use the proof of Step 1.

From (6.2) and (6.4)-(6.6), we have

$$\begin{aligned} i(c_j^{2m_j})+\nu(c_j^{2m_j}) &\le i(c_j^{2m_j+1})-i(c_j)+\frac{e(P_{c_j})}{2}, &(6.32)\\ &= 2N+\frac{e(P_{c_j})}{2},\quad 1\le j\le p. &(6.33)\end{aligned}$$

Now if c_j does not satisfy all of (ii)-(iv), then by the proof of Theorem 2.2 in [LoZ1], The strict inequality in (6.32) must hold. In fact, (ii) follows from Cases 1, 3, 7 and (iii) follows from Case 8 of Theorem 2.2 in [LoZ1] respectively. Hence we only need to check the case (iv). As in the proof of Theorem 2.2 in [LoZ1], it suffices to show that

$$\nu(\gamma,m)-\frac{e(M)}{2}<i(\gamma,m+1)-i(\gamma,m)-i(\gamma,1),\quad \forall m\in\mathbf{N}, \tag{6.34}$$

when $M=\gamma(\tau)=R(\theta)\diamond R(2\pi-\theta)$ with $\frac{\theta}{\pi}\notin\mathbf{Q}$. By Case 6 in P.336 of [LoZ1] and the symplectic additivity of indices and nullities, we have

$$\nu(\gamma,m)-\frac{e(M)}{2}=2-2\varphi\left(\frac{m\theta}{2\pi}\right)-2\varphi\left(\frac{m(2\pi-\theta)}{2\pi}\right)=-2. \tag{6.35}$$

On the other hand, we have

$$\begin{aligned} & i(\gamma,m+1)-i(\gamma,m)-i(\gamma,1)\\ =\;& 2\left(E\left(\frac{(m+1)\theta}{2\pi}\right)+E\left(\frac{(m+1)(2\pi-\theta)}{2\pi}\right)\right)\\ & -2\left(E\left(\frac{m\theta}{2\pi}\right)+E\left(\frac{m(2\pi-\theta)}{2\pi}\right)\right)-2\\ =\;& 0. \end{aligned}\tag{6.36}$$

In the last equality, we have used the fact that $E(a)+E(-a)=1$ whenever $a\in(0,+\infty)\setminus\mathbf{Z}$. Hence (6.34) is true. Now (6.9)-(6.11) still hold. Hence the same proof as in Step 1 yields a contradiction.

We then show that there must exist d satisfying (i)-(v). Suppose none of the closed geodesics in $\{c_j\}_{1\le j\le p}$ satisfies all of (i)-(v). Then it is easy to see that (6.15), (6.19)-(6.23) still hold. In fact, we only need to check (6.22). We have

$$\overline{C}_q(E,\,c_j^m)=0,\qquad \forall m\le 2m_j,\;\; q\ge 2N+n-1,\;\; 1\le j\le p,$$

which follows easily from Propositiom 2.1, (6.2) and (11.2.4) in Theorem 11.2.1 of [Lon2]

$$i(c_j^{2m_j-1}) + \nu(c_j^{2m_j-1}) = 2N - (i(c_j) + 2S^+_{P_{c_j}}(1) - \nu(c_j)) \leq 2N,$$

where we have used (6.1) and the fact that $2S^+_{P_{c_j}}(1) - \nu(c_j) \geq -(n-1)$, which follows from (15.4.21) in p.340 of [Lon2]. This yields (6.22).

Thus the same proof as in Step 1 yields a contradiction.

At last we prove that there must exist d satisfying (i)-(vi). Suppose none of the closed geodesics in $\{c_j\}_{1\leq j\leq p}$ satisfies all of (i)-(vi). Then by the above argument, we assume c_j for some $1 \leq j \leq p$ satisfies all (i)-(v) but not (vi). We consider $g = c_j^{T(c_j)}$. Then by (i)-(iii) and the assumption, P_g can be connected in $\Omega^0(P_g)$ to $I_{2p'_0} \diamond N_1(1,\,-1)^{\diamond p'_+}$ with $p'_0 + p'_+ = n-1$ as in Theorem 4.1. In fact, by (i)-(iii) in Claim 3 and the assumption, the basic normal form decomposition (4.1) in Theorem 4.1 becomes

$$\begin{aligned} M_{c_j} = \quad & I_{2p_0} \diamond N_1(1,\,-1)^{\diamond p_+} \diamond N_1(-1,\,1)^{\diamond q_-} \diamond (-I_{2q_0}) \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\lambda_1,\,v_1) \diamond \cdots \diamond N_2(\lambda_{r_0},\,v_{r_0}) \end{aligned}$$

together with $\frac{\theta_j}{\pi} \in \mathbf{Q}$ for $1 \leq j \leq r$, $\frac{\beta_j}{\pi} \in \mathbf{Q}$ for $1 \leq j \leq r_0$ and $p_0 + p_+ + q_- + q_0 + r + 2r_0 = n-1$. By Lemma 5.2, we have $\frac{T(c_j)\theta_j}{2\pi} \in \mathbf{Z}$, $\frac{T(c_j)\beta_j}{2\pi} \in \mathbf{Z}$ and $2|T(c_j)$ whenever $-1 \in \sigma(M_{c_j})$. Hence $R(\theta_j)^{T(c_j)} = I_2$, $N_2(\lambda_j,\,v_j)^{T(c_j)}$ can be connected within $\Omega^0(N_2(\lambda_j,\,v_j)^{T(c_j)})$ to $N_1(1,\,-1)^{\diamond 2}$ and $(-I_2)^{T(c_j)} = I_2$, $N_1(-1,\,1)^{T(c_j)}$ can be connected within $\Omega^0(N_1(-1,\,1)^{T(c_j)})$ to $N_1(1,\,-1)$ whenever $-1 \in \sigma(M_{c_j})$. Thus $p'_0 = p_0 + q_0 + r$ and $p'_+ = p_+ + q_- + 2r_0$ and then P_g behaves as claimed.

Now by Theorem 4.1, we have

$$i(g^m) = m(i(g) + p'_0) - p'_0, \quad \nu(g^m) \equiv 2p'_0 + p'_+. \qquad \forall m \in \mathbf{N}. \tag{6.37}$$

Hence

$$i(g^m) + \nu(g^m) = m(i(g) + p'_0) + p'_0 + p'_+. \qquad \forall m \in \mathbf{N}. \tag{6.38}$$

On the other hand

$$\begin{aligned} & m(i(g) + \nu(g)) - (n-1)(m-1) \\ = \quad & m(i(g) + 2p'_0 + p'_+) - (p'_0 + p'_+)(m-1) \\ = \quad & m(i(g) + p'_0) + p'_0 + p'_+. \qquad \forall m \in \mathbf{N}. \end{aligned} \tag{6.39}$$

By Lemma 2.4 and (v), we have

$$k_{\nu(g)}(g) = k_{\nu(c_j^{T(c_j)})}(c_j^{T(c_j)}) \geq k^{\beta(c_j^{T(c_j)})}_{\nu(c_j^{T(c_j)})}(c_j^{T(c_j)}) \neq 0. \tag{6.40}$$

Hence we can use Theorem 4.2 to obtain infinitely many prime closed geodesics, which contradicts to the assumption (F). This complete the proof of Step 2. ∎

Proof of Theorem 1.2. Suppose $d \in \{c_j\}_{1\le j\le p}$ is a closed geodesic satisfying (i)-(vi) of Claim 3 above. Then we have the following

$$i(d^m) + \nu(d^m) - i(d) - \nu(d) \in 2\mathbf{Z}, \quad \forall m \in \mathbf{N}. \tag{6.41}$$

One can prove this by verifying each basic norm form in the decomposition of M_d, then using the symplectic additivity of indices and nullities to obtain (6.41). Here we omit the details.

By (v), we have $k_{\nu(d^{T(d)})}^{\beta(d^{T(d)})}(d^{T(d)}) \neq 0$. Hence by Lemma 2.4 and Proposition 2.7, we have

$$\begin{aligned}
k_{\nu(d^{T(d)})}(d^{T(d)}) &\ge k_{\nu(d^{T(d)})}^{\beta(d^{T(d)})}(d^{T(d)}) = 1, & (6.42)\\
0 \le k_{\nu(d^m)}^{\beta(d^m)}(d^m) &\le k_{\nu(d^m)}(d^m) = 1, \qquad \forall m \in \mathbf{N}, & (6.43)\\
0 \le k_l^{\beta(d^m)}(d^m) &\le k_l(d^m) = 0, \qquad \forall 0 \le l \le \nu(d^m) - 1,\ m \in \mathbf{N}. & (6.44)
\end{aligned}$$

Note that in order to get (6.43) and (6.44), we have used the same argument as in the last paragraph in the proof of Lemma 5.2.

By (5.8), we have

$$\begin{aligned}
\hat{\chi}(d) &= \frac{1}{T(d)} \sum_{\substack{1\le m\le T(d)\\ 0\le l\le 2n-2}} (-1)^{i(d^m)+l} k_l^{\beta(d^m)}(d^m) & (6.45)\\
&= \frac{1}{T(d)} \sum_{1\le m\le T(d)} (-1)^{i(d^m)+\nu(d^m)} k_{\nu(d^m)}^{\beta(d^m)}(d^m) & (6.46)\\
&= \frac{(-1)^{i(d)+\nu(d)}}{T(d)} \sum_{1\le m\le T(d)} k_{\nu(d^m)}^{\beta(d^m)}(d^m) & (6.47)\\
&\neq 0. & (6.48)
\end{aligned}$$

Here to get (6.46), we have used (6.44). To get (6.47), we have used (6.41). To get (6.48), we have used (6.42) and (6.43). Note that $B(n,1) \in \mathbf{Q}$. Hence the theorem follows easily from (vi) of Claim 3, Theorem 4.1 and Theorem 5.5. ∎

Proof of Theorem 1.4. Suppose $d \in \{c_j\}_{1\le j\le p}$ is a closed geodesic satisfying (i)-(vi) of Claim 3 above. Suppose $\omega_1^{\pm 1}, \ldots, \omega_r^{\pm 1}$ with $\omega_i = e^{\sqrt{-1}\theta_i}$ are those eigenvalues of P_d that satisfy $\frac{\theta_i}{\pi} \notin \mathbf{Q}$ for $1 \le i \le r$. Then we have

Claim 4. *There exists* $M \in \mathrm{Sp}(2n-2)$ *such that*

$$MP_dM^{-1} = R(\widehat{\theta}_1) \diamond \cdots \diamond R(\widehat{\theta}_r) \diamond M_0, \tag{6.49}$$

with $\widehat{\theta}_i = \theta_i$ *or* $2\pi - \theta_i$ *for* $1 \le i \le r$ *and* $\sigma(M_0) \subset \mathbf{U} \cap \{e^{\sqrt{-1}\theta} \mid \frac{\theta}{\pi} \in \mathbf{Q}\}$.

In fact, by Theorem 1.6.11 of [Lon2], we have $M_1 \in \mathrm{Sp}(2n-2)$ such that

$$M_1 P_d M_1^{-1} = S_1 \diamond \cdots \diamond S_{m_1} \diamond S_0, \tag{6.50}$$

with $S_0 \in \mathrm{Sp}(2k_0)$ with $k_0 \geq 0$ and $\omega_1 \notin \sigma(S_0)$, $k_i \geq 1$ and $S_i \in \mathrm{Sp}(2k_i)$ is of the normal form $N_{k_i}(\lambda_i, b_i)$ with $\lambda_i = \omega_1$ or ω_1^{-1} defined in Section 1.6 of [Lon2].

Then by Case 3 and 4 in Section 1.8 of [Lon2], if $k_i \geq 3$ for some $1 \leq i \leq m_1$, then S_i can be connected within $\Omega^0(S_i)$ to $\widetilde{S_i}$ with $e(\widetilde{S_i}) < e(S_i)$. This contradicts to (i) of Claim 3. Hence $S_i = N_{l_i}(\lambda_i, b_i)$ for $l_i = 1$ or 2 and $1 \leq i \leq m_1$. We next prove that $l_i = 1$ for $1 \leq i \leq m_1$. Suppose $l_i = 2$ for some i. Then by (ii) and (iii) of Claim 3, S_i is not a basic normal form, hence $S_i \in \mathcal{M}_{\omega_1}^2(4)$. By Case 4 of [Lon2], S_i can be connected within $\Omega^0(S_i)$ to $R(\omega_1) \diamond R(2\pi - \omega_1)$. This contradicts to (iv) of Claim 3. Now (6.50) becomes

$$M_1 P_d M_1^{-1} = R(\widehat{\theta}_1)^{\diamond m_1} \diamond S_0, \tag{6.51}$$

with $\widehat{\theta_1} = \theta_1$ or $2\pi - \theta_1$. We continue the above argument for at most r times and then obtain (6.49). This proves Claim 4.

At last, let $g = d^{T(d)}$. Then it is easy to see that g is of elliptic-parabolic type. In fact, we have

$$MP_{d^{T(d)}}M^{-1} = (MP_d M^{-1})^{T(d)} = R(T(d)\widehat{\theta}_1) \diamond \cdots \diamond R(T(d)\widehat{\theta}_r) \diamond M_0^{T(d)}$$

while by the proof Lemma 5.2, all the eigenvalues of $M_0^{T(d)}$ equal to 1. Hence Theorem 1.4 holds by the definition of elliptic-parabolic type. ∎

7 Existence of three closed geodesics on $(S^3,\, F)$

In this section, we give a proof of Theorem 1.5 based on the results in the previous sections.

Firstly by Lemma 2 of [Rad4], we have the following

Lemma 7.1. *Suppose the Finsler 3-sphere $(S^3,\, F)$ with reversibility λ and flag curvature K satisfies $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$. Then every closed geodesic c on $(S^3,\, F)$ has mean index $\hat{i}(c) > 2$.*

Remark 7.2. Note that by Theorem 4.1, the index iteration formulae of $N_1(1,\, 1)$ and I_2 coincide and both can be viewed as a rotation matrix $R(\theta)$ with $\theta = 2\pi$. Similarly $N_1(-1,\, -1)$ and $-I_2$ can be viewed as a rotation matrix $R(\theta)$ with $\theta = \pi$. Hence in the following, we only consider the case $R(\theta)$ with $\theta \in (0,\, 2\pi]$, the same proof still works for $N_1(1,\, 1)$ and $N_1(-1,\, -1)$.

Lemma 7.3. *Under the assumption of Lemma 7.1, suppose P_c can be connected within $\Omega^0(P_c)$ to M_c as in Theorem 4.1 with $M_c = R(\theta_1) \diamond R(\theta_2)$ with $\theta_j \in (0,\, 2\pi]$ for $j = 1, 2$. Then we have the*

following

$$i(c^{m+q}) \geq i(c^m) + 2, \qquad \forall m \in \mathbf{N},\ q \geq 2. \tag{7.1}$$

Proof. By Theorem 4.1, we have

$$i(c^m) = m(i(c) - 2) + 2\mathcal{E}\left(\frac{m\theta_1}{2\pi}\right) + 2\mathcal{E}\left(\frac{m\theta_2}{2\pi}\right) - 2, \quad \forall m \in \mathbf{N}, \tag{7.2}$$

with $\theta_j \in (0,\ 2\pi]$ for $j = 1, 2$ and $i(c) \in 2\mathbf{N}$.

Now if $i(c) \geq 4$, the lemma is obvious by (7.2).

Next consider the case $i(c) = 2$. We have

$$i(c^m) = 2\mathcal{E}\left(\frac{m\theta_1}{2\pi}\right) + 2\mathcal{E}\left(\frac{m\theta_2}{2\pi}\right) - 2, \quad \forall m \in \mathbf{N}, \tag{7.3}$$

By Lemma 7.1, we have

$$\hat{i}(c) = \frac{\theta_1}{\pi} + \frac{\theta_2}{\pi} > 2.$$

Hence with out loss of generality, we may assume $\frac{\theta_1}{\pi} > 1$. Thus we have

$$\begin{aligned} i(c^{m+p}) &= 2\mathcal{E}\left(\frac{(m+p)\theta_1}{2\pi}\right) + 2\mathcal{E}\left(\frac{(m+p)\theta_2}{2\pi}\right) - 2 \\ &\geq 2\mathcal{E}\left(\frac{m\theta_1}{2\pi} + \frac{\theta_1}{\pi}\right) + 2\mathcal{E}\left(\frac{m\theta_2}{2\pi}\right) - 2 \\ &\geq 2\mathcal{E}\left(\frac{m\theta_1}{2\pi}\right) + 2\mathcal{E}\left(\frac{m\theta_2}{2\pi}\right), \end{aligned}$$

for $p \geq 2$. Note that in the second inequality above, we have used the fact that $\mathcal{E}(a+b) \geq \mathcal{E}(a) + 1$ for any $a \in \mathbf{R}$ and $b > 1$. Hence the lemma holds. ∎

Lemma 7.4. *Under the assumption of Lemma 7.1, suppose P_c can be connected to M_c as in Theorem 4.1 with $M_c = R(\theta_1) \diamond R(\theta_2)$ with $\frac{\theta_j}{\pi} \in (0,\ 2] \cap \mathbf{Q}$ for $j = 1, 2$. Let $m \in \mathbf{N}$ satisfy $\nu(c^m) = 4$. Then we have the following*

$$i(c^{m-q}) + \nu(c^{m-q}) \leq i(c^m) - 2, \qquad \forall q \geq 2. \tag{7.4}$$

Proof. As in Lemma 7.3, we have (7.2) with

$$\frac{\theta_j}{2\pi} = \frac{r_j}{s_j}, \quad r_j,\ s_j \in \mathbf{N}, \ \ (r_j,\ s_j) = 1, \ \ j = 1, 2. \tag{7.5}$$

By (6.2), it suffices to prove the case $q = 2$.

Note that $\nu(c^m) = 4$ implies that $s_j | m$, i.e., s_j is a factor of m for $j = 1,\ 2$. We may assume $s_1 \leq s_2$ without loss of generality. Hence we have

$$\nu(c^{m-1}) = \nu(c), \qquad \nu(c^{m-2}) = \nu(c^2). \tag{7.6}$$

In fact, $\nu(c^{m-1}) = 2k$ for $k \in \{0,\,1,\,2\}$ if and only if $s_j|m-1$ for $1 \le j \le k$, and this is equivalent to $s_j|1$ for $1 \le j \le k$, and this implies $\nu(c) = 2k$. Similarly, $\nu(c^{m-2}) = 2k$ for $k \in \{0,\,1,\,2\}$ if and only if $s_j|m-2$ for $1 \le j \le k$, and this is equivalent to $s_j|2$ for $1 \le j \le k$, and this implies $\nu(c^2) = 2k$.

Now if $\nu(c) \ge 2$, then by (6.2) and (7.6), we have

$$i(c^{m-2}) + \nu(c^{m-2}) \le i(c^{m-1}) \le i(c^m) - \nu(c^{m-1}) \le i(c^m) - 2.$$

Hence the lemma holds.

Next consider the case $\nu(c) = 0$. If $\nu(c^2) = 0$, then by Lemma 7.3 and (7.6), we have

$$i(c^{m-2}) + \nu(c^{m-2}) = i(c^{m-2}) \le i(c^m) - 2.$$

Hence it remains to consider the case $\nu(c^2) \ge 2$ and $\nu(c) = 0$. This implies $s_1 = 2$ and then $r_1 = 1$ by (7.5). Now we have

$$\begin{aligned} i(c^m) &= m(i(c)-2) + 2\mathcal{E}\left(\frac{m}{2}\right) + 2\mathcal{E}\left(\frac{m\theta_2}{2\pi}\right) - 2 \\ &= m(i(c)-2) + 2\mathcal{E}\left(\frac{m-2}{2}\right) + 2\mathcal{E}\left(\frac{(m-2)\theta_2}{2\pi} + \frac{\theta_2}{\pi}\right) \\ &= i(c^{m-2}) + 2 + 2(i(c)-2) + 2\left(\mathcal{E}\left(\frac{(m-2)\theta_2}{2\pi} + \frac{\theta_2}{\pi}\right) - \mathcal{E}\left(\frac{(m-2)\theta_2}{2\pi}\right)\right). \end{aligned} \tag{7.7}$$

Now if $i(c) \ge 4$, then (7.7) implies that $i(c^m) \ge i(c^{m-2}) + 6$. This together with $\nu(c^{m-2}) \le 4$ prove the lemma.

If $i(c) = 2$, then by Lemma 7.1, we have $\hat{i}(c) = 1 + \frac{\theta_2}{\pi} > 2$. This yields $\frac{\theta_2}{\pi} > 1$, and then $s_2 \ge 3$. Hence $\nu(c^{m-2}) \le 2$ by (7.6). Now the last term in (7.7) is not less than 2. Hence

$$i(c^{m-2}) + \nu(c^{m-2}) \le i(c^m) - 4 + 2 = i(c^m) - 2.$$

This proves the whole lemma. ∎

Lemma 7.5. *Let M_j and b_j be the integers defined at the beginning of Section 6. If $M_k = b_k$ for some $k \in \mathbf{N}_0$, then we have*

$$M_k - M_{k-1} + \cdots + (-1)^k M_0 = b_k - b_{k-1} + \cdots + (-1)^k b_0, \tag{7.8}$$

$$M_{k-1} - M_{k-2} + \cdots + (-1)^{k-1} M_0 = b_{k-1} - b_{k-2} + \cdots + (-1)^{k-1} b_0, \tag{7.9}$$

Proof. By Theorem 3.2, we have

$$\begin{aligned} M_k - M_{k-1} + \cdots + (-1)^k M_0 &\ge b_k - b_{k-1} + \cdots + (-1)^k b_0, \\ M_{k-1} - M_{k-2} + \cdots + (-1)^{k-1} M_0 &\ge b_{k-1} - b_{k-2} + \cdots + (-1)^{k-1} b_0, \end{aligned}$$

These together with $M_k = b_k$ yields (7.9) and then (7.8). ∎

Lemma 7.6. *Under the assumption of Lemma 7.1, there are at least two prime closed geodesics on (S^3, F). If there are exactly two prime closed geodesics, then at least one of them has Poincaré map $P_c = R(\theta_1) \diamond R(\theta_2)$ with $\frac{\theta_j}{\pi} \in (0, 2] \setminus \mathbf{Q}$ for $j = 1, 2$ in an appropriate coordinates.*

Proof. By [Fet1], there exists at least one closed geodesic on (S^3, F). Now we prove $p \geq 2$, where p is the integer in the assumption (F). As in the proof of Theorem 1.1 in Section 6, by Theorem 11.2.1 of [Lon2], we obtain some $(N, m_1, \dots, m_p) \in \mathbf{N}^{p+1}$ such that

$$\begin{aligned}
i(c_j^{2m_j}) &\geq 2N - \frac{e(P_{c_j})}{2}, &(7.10)\\
i(c_j^{2m_j}) + \nu(c_j^{2m_j}) &\leq 2N + \frac{e(P_{c_j})}{2}, &(7.11)\\
i(c_j^{2m_j-m}) + \nu(c_j^{2m_j-m}) &\leq 2N - (i(c_j) + 2S^+_{P_{c_j}}(1) - \nu(c_j)), \quad \forall m \in \mathbf{N}. &(7.12)\\
i(c_j^{2m_j+m}) &\geq 2N + i(c_j), \quad \forall m \in \mathbf{N}. &(7.13)\\
\nu(c_j^{2m_j+1}) &= \nu(c_j), &(7.14)
\end{aligned}$$

By the proof of Theorem 1.1, we may assume c_1 satisfies

$$M_{c_1} = R(\theta_1) \diamond Q_1, \quad i(c_1^{2m_1}) + \nu(c_1^{2m_1}) = 2N + 2, \quad k^{\beta(c_1^{2m_1})}_{\nu(c_1^{2m_1})}(c_1^{2m_1}) \neq 0, \qquad (7.15)$$

for some $\frac{\theta_1}{\pi} \in \mathbf{R} \setminus \mathbf{Q}$ and $Q_1 \in Sp(2)$. Here we use notations as in the proof of Theorem 1.1. In fact, by the proof of Theorem 1.1, we must have $\overline{C}_{2N+2}(E, c_j^{2m_j}) \neq 0$ and M_{c_j} contain a term $R(\theta)$ with $\frac{\theta}{\pi} \notin \mathbf{Q}$ for some $1 \leq j \leq p$. Thus (7.15) follows from Proposition 2.1.

By p.340 of [Lon2], we have

$$\begin{aligned}
&2S^+_{P_{c_1}}(1) - \nu(c_1)\\
&\quad = 2S^+_{R(\theta_1)}(1) - \nu_1(R(\theta_1)) + 2S^+_{Q_1}(1) - \nu_1(Q_1)\\
&\quad = 0 + 2S^+_{Q_1}(1) - \nu_1(Q_1)\\
&\quad \geq -1. \qquad (7.16)
\end{aligned}$$

Now by (6.1) we have

$$\begin{aligned}
i(c_1^{2m_1-m}) + \nu(c_1^{2m_1-m}) &\leq 2N - 1, \quad \forall m \in \mathbf{N}. &(7.17)\\
i(c_1^{2m_1+m}) &\geq 2N + 2, \quad \forall m \in \mathbf{N}. &(7.18)
\end{aligned}$$

Now (7.15), (7.17)-(7.18) together with Propositions 2.1 and 2.6 yield

$$\overline{C}_{2N}(E, c_1^m) = 0, \quad \forall m \in \mathbf{N}. \qquad (7.19)$$

By Theorems 3.1 and 3.2 we have

$$M_{2N} \geq b_{2N} = 2. \tag{7.20}$$

Hence $p \geq 2$ holds. This proves the first part of the lemma.

Next we assume $p = 2$. Then we have

$$\overline{C}_{2N}(E, c_2^q) \neq 0, \tag{7.21}$$

for some $q \in \mathbf{N}$ by (7.20).

In order to prove the second part of the lemma, we need the following

Claim. The only possible value of q that satisfies(7.21) is $2m_2$.

In fact, again by p.340 of [Lon2], we have

$$2S^+_{P_{c_2}}(1) - \nu(c_2) \geq -2,$$

where the equality holds if and only if $M_{c_2} = N_1(1, -1)^{\diamond 2}$. Now by (6.1), (7.12) and (7.13), we have

$$i(c_2^{2m_2-m}) + \nu(c_2^{2m_2-m}) \leq 2N, \quad \forall m \in \mathbf{N}. \tag{7.22}$$

$$i(c_2^{2m_2+m}) \geq 2N+2, \quad \forall m \in \mathbf{N}. \tag{7.23}$$

Here the equality in (7.22) holds if and only if $m = 1$, $M_{c_2} = N_1(1, -1)^{\diamond 2}$ and $i(c_2) = 2$ hold simultaneously.

If the claim is not true, then by (7.21) and Proposition 2.1, the equality for $m = 1$ in (7.22) must hold. Thus by Theorem 4.1, we have

$$i(c_2^m) = 2m, \quad \nu(c_2^m) = 2, \qquad \forall m \in \mathbf{N}.$$

Hence by Proposition 2.1 again, we have

$$\overline{C}_{2N}(E, c_2^{2m_2-1}) = \mathbf{Q}, \quad \overline{C}_{2N}(E, c_2^m) = 0, \ \ \forall m \neq 2m_2 - 1.$$

This together with (7.19) yield

$$1 = M_{2N} \geq b_{2N} = 2.$$

This contradiction proves the claim.

Now we prove the second part of the lemma by contradiction, i.e., suppose $Q_1 \neq R(\theta_2)$ with some $\frac{\theta_2}{\pi} \in (0, 2] \setminus \mathbf{Q}$ in the decomposition (7.15). Then by Theorem 1.2, we must have

$$M_{c_2} = R(\theta') \diamond Q', \tag{7.24}$$

for some $\frac{\theta'}{\pi} \in \mathbf{R} \setminus \mathbf{Q}$ and $Q' \in Sp(2)$.

Now (7.19)-(7.21) imply that

$$M_{2N} = \dim \overline{C}_{2N}(E, c_2^{2m_2}) \geq 2.$$

Note that by (7.24), we have $\nu(c_2^{2m_2}) \leq 2$. If $\nu(c_2^{2m_2}) = 2$, then by Theorem 4.1, we must have $Q' = R(\vartheta)$ for some $\frac{\vartheta}{\pi} \in (0, 2] \cap \mathbf{Q}$. Thus it follows from (4.3) that $i(c_2^m) \in 2\mathbf{Z}$ for all $m \in \mathbf{N}$. Hence either $2N = i(c_2^{2m_2})$ or $2N = i(c_2^{2m_2}) + 2$ holds. Now by Definition 2.3 for $\nu(c_2^{2m_2}) \leq 2$, we have

$$\overline{C}_{2N}(E, c_2^{2m_2}) = \mathbf{Q}.$$

This contradiction proves the lemma. ∎

Now we can give the proof of Theorem 1.5.

Proof of Theorem 1.5. We prove the theorem by contradiction, i.e., by Lemma 7.6, we assume $p = 2$ in the assumption (F). By Lemma 7.6, we may assume $P_{c_1} = R(\theta_1) \diamond R(\theta_2)$ with $\frac{\theta_j}{\pi} \in (0, 2] \setminus \mathbf{Q}$ for $j = 1, 2$ and (7.15) holds. Assume the linearized Poincaré map P_{c_2} of c_2 can be connected to M_{c_2} in Theorem 4.1. Then due to Remark 7.2 and Theorem 4.1, we have the following cases according to M_{c_2}.

Case 1. *We have $M_{c_2} = R(\vartheta_1) \diamond R(\vartheta_2)$ with $\frac{\vartheta_j}{\pi} \in (0, 2] \cap \mathbf{Q}$ for $j = 1, 2$.*

By p.340 of [Lon2], we have

$$2S^+_{P_{c_j}}(1) - \nu(c_j) \geq 0, \qquad j = 1,\, 2. \tag{7.25}$$

Now by (6.1), (7.12)-(7.13), we have

$$\begin{aligned} i(c_j^{2m_j - m}) + \nu(c_j^{2m_j - m}) &\leq 2N - 2, \quad \forall m \in \mathbf{N}, \qquad (7.26)\\ i(c_j^{2m_j + m}) &\geq 2N + 2, \quad \forall m \in \mathbf{N}, \qquad (7.27) \end{aligned}$$

for $j = 1,\, 2$. Note that by the definition of $2m_2$, we have $2m_2 \frac{\vartheta_j}{2\pi} \in \mathbf{Z}$ for $j = 1,\, 2$. Thus we have

$$\nu(c_2^{2m_2}) = 4, \qquad i(c_2^{2m_2}) = 2N - 2, \tag{7.28}$$

where the latter holds by (7.10)-(7.11). By (7.2), we have for $m \geq 2$

$$\begin{aligned} i(c_2^{2m_2 + m}) &\geq i(c_2^{2m_2}) + 2(i(c_2) - 2) + 2\mathcal{E}\left(\frac{\vartheta_1}{\pi}\right) + 2\mathcal{E}\left(\frac{\vartheta_2}{\pi}\right)\\ &\geq i(c_2^{2m_2}) + 6\\ &= 2N + 4, \qquad \forall m \geq 2. \qquad (7.29) \end{aligned}$$

Where in the first inequality, we have used $2m_2\frac{\vartheta_j}{2\pi}\in\mathbf{Z}$ for $j=1,\,2$. The second inequality follows by $\hat{i}(c_2)>2$ as before. Note that by (7.21) and (iii) of Proposition 2.6, we have

$$\overline{C}_{2N-2}(E,\ c_2^{2m_2})=0=\overline{C}_{2N+2}(E,\ c_2^{2m_2}). \tag{7.30}$$

Hence (7.26)-(7.30) imply

$$\sum_{m\in\mathbf{N}}\dim\overline{C}_{2N+2}(E,\ c_2^m)\le 1. \tag{7.31}$$

with equality holds if and only if $i(c_2)=2$ and $k_0^{+1}(c_2^{2m_2+1})=k_0^{+1}(c_2)=1$. Note that here we have used $T(c_2)|2m_2$.

By (7.15) and Lemma 7.3, we have

$$1\le\sum_{m\in\mathbf{N}}\dim\overline{C}_{2N+2}(E,\ c_1^m)\le 2. \tag{7.32}$$

where the second equality holds if and only if $i(c_1)=2$.

By Theorem 3.2 and (7.31)-(7.32), we have

$$2=b_{2N+2}\le M_{2N+2}=\sum_{m\in\mathbf{N},\ 1\le j\le 2}\dim\overline{C}_{2N+2}(E,\ c_j^m)\le 3.$$

Now if $M_{2N+2}=3$, by (7.31)-(7.32), we have

$$\begin{aligned}
&\overline{C}_q(E,\ c_1^m)=0=\overline{C}_q(E,\ c_2^m), \qquad \forall m\in\mathbf{N},\ q\le 1,\\
&\overline{C}_2(E,\ c_1)=\mathbf{Q}=\overline{C}_2(E,\ c_2),\\
&\overline{C}_2(E,\ c_1^m)=0=\overline{C}_2(E,\ c_2^m), \quad \forall m\ge 2,\\
&\overline{C}_3(E,\ c_1^m)=0=\overline{C}_3(E,\ c_2^m), \quad \forall m\in\mathbf{N}.
\end{aligned}$$

Here the latter two hold by Propositions 2.1 and 2.6 together with $i(c_j^m)\ge 4$ for $m\ge 2$ and $j=1,\,2$, which follows from $\hat{i}(c_j)>2$ for $j=1,\,2$ as before. Hence by Theorems 3.1 and 3.2, we have

$$-2=M_3-M_2+M_1-M_0\ge b_3-b_2+b_1-b_0=-1.$$

This contradiction proves that

$$M_{2N+2}=b_{2N+2}=2. \tag{7.33}$$

By (7.28), Lemma 7.4, Proposition 2.1 and (7.30), we have

$$\sum_{m\in\mathbf{N}}\dim\overline{C}_{2N-2}(E,\ c_2^m)\le 1. \tag{7.34}$$

with equality holds if and only if $i(c_2)=2$ and $k^{+1}_{\nu(c_2^{2m_2-1})}(c_2^{2m_2-1})=k^{+1}_{\nu(c_2)}(c_2)=1$. Note that here we have used $T(c_2)|2m_2$. By Lemma 7.3, we have

$$\sum_{m\in\mathbf{N}}\dim\overline{C}_{2N-2}(E,\ c_1^m)\le 2. \tag{7.35}$$

By Theorem 3.2 and (7.34)-(7.35), we have

$$2=b_{2N-2}\le M_{2N-2}=\sum_{m\in\mathbf{N},\ 1\le j\le 2}\dim\overline{C}_{2N-2}(E,\ c_j^m)\le 3. \tag{7.36}$$

Next we have two subcases according to the value of M_{2N-2}.

Subcase 1.1. $M_{2N-2}=3$.

In this subcase, the equality in (7.34) and (7.35) must hold. In particular, we have

$$\overline{C}_{2N-3}(E,\ c_j^m)=0,\qquad \forall m\in\mathbf{N},\ \ j=1,\,2. \tag{7.37}$$

In fact, since c_1^m is non-degenerate and $i(c_1^m)$ is even for all $m\in\mathbf{N}$ by (7.2). Hence (7.37) holds for c_1^m by Proposition 2.1. By (7.28), Lemma 7.4 and Proposition 2.1, (7.37) holds for c_2^m with $m\le 2m_2-2$. By (7.27)-(7.28) and Proposition 2.1, (7.37) holds for c_2^m with $m\ge 2m_2$. Since the equality in (7.34) holds, we have

$$i(c_2^{2m_2-1})+\nu(c_2^{2m_2-1})=2N-2,\qquad k^{+1}_{\nu(c_2^{2m_2-1})}(c_2^{2m_2-1})=1.$$

Hence (7.37) holds for $c_2^{2m_2-1}$ by Propositions 2.1 and 2.6.

Now we have the following diagram.

$$\begin{array}{c|cccccc} & 2N-3 & 2N-2 & 2N-1 & 2N & 2N+1 & 2N+2\\ \hline M_{*,\,1} & 0 & 2 & 0 & 0 & 0 & \xi\\ M_{*,\,2} & 0 & 1 & k_1 & k_2 & k_3 & \zeta \end{array} \tag{7.38}$$

where $M_{*,\,j}=\sum_{m\in\mathbf{N}}\dim\overline{C}_*(E,\ c_j^m)$ for $j=1,2$, $k_l=k_l^{+1}(c_2^{2m_2})$ for $1\le l\le 3$ and $\xi+\zeta=2$. In fact, the first column follows from (7.37). The second column follows from (7.34)-(7.35). The first row follows from (7.15) and (7.26)-(7.27). The second row follows from (7.26)-(7.27). $\xi+\zeta=2$ follows from (7.33).

Since $M_{2N-3}=b_{2N-3}=0$ and $M_{2N+2}=b_{2N+2}=2$ hold by the first and last column of (7.38), we have by Lemma 7.5

$$M_{2N+2}-M_{2N+1}+\cdots-M_1+M_0 \;=\; b_{2N+2}-b_{2N+1}+\cdots-b_1+b_0, \tag{7.39}$$

$$M_{2N-3}-M_{2N-4}+\cdots+M_1-M_0 \;=\; b_{2N-3}-b_{2N-4}+\cdots+b_1-b_0, \tag{7.40}$$

Subtracting (7.40) from (7.39) and using Theorem 3.1 together with (7.38), we have

$$\begin{aligned} 2-k_3+k_2-k_1+3 &= M_{2N+2}-M_{2N+1}+M_{2N}-M_{2N-1}+M_{2N-2} \\ &= b_{2N+2}-b_{2N+1}+b_{2N}-b_{2N-1}+b_{2N-2}=6. \end{aligned}$$

This implies

$$k_2-k_1-k_3=1. \tag{7.41}$$

Since c_1^m is non-degenerate and $i(c_1^m)$ is even for all $m\in\mathbf{N}$, we have $T(c_1)=1$ by Lemma 5.2. Hence we have $\hat{\chi}(c_1)=1$ by (5.8). Now by Lemma 7.1, we have

$$\frac{\hat{\chi}(c_1)}{\hat{i}(c_1)}<\frac{1}{2}. \tag{7.42}$$

Since $T(c_2)|2m_2$, we have $k_l^{+1}(c_2^{T(c_2)})=k_l^{+1}(c_2^{2m_2})$ for all $l\in\mathbf{N}$. This together with $i(c_2^m)$ is even for all $m\in\mathbf{N}$ and (7.41), we have

$$\chi(c_2^{T(c_2)})=k_2-k_1-k_3=1. \tag{7.43}$$

On the other hand, we have

$$\chi(c_2^m)\le 1, \qquad \forall m<T(c_2). \tag{7.44}$$

In fact, this follows from $i(c_2^m)$ is even, $\nu(c_2^m)\le 2$ for all $m<T(c_2)$, Proposition 2.6 and (5.7),. Now by (5.8), we have

$$\hat{\chi}(c_2)=\frac{1}{T(c_2)}\sum_{1\le m\le T(c_2)}\chi(c_2^m)\le\frac{1}{T(c_2)}T(c_2)=1.$$

Now by Lemma 7.1, we have

$$\frac{\hat{\chi}(c_2)}{\hat{i}(c_2)}<\frac{1}{2}. \tag{7.45}$$

By Theorem 5.5, (5.11), (7.42) and (7.45), we have

$$1=B(3,\,1)=\frac{\hat{\chi}(c_1)}{\hat{i}(c_1)}+\frac{\hat{\chi}(c_2)}{\hat{i}(c_2)}<\frac{1}{2}+\frac{1}{2}=1. \tag{7.46}$$

This contradiction proves the theorem in this subcase.

Subcase 1.2. $M_{2N-2}=2$.

In this subcase, we have the following diagram.

$$\begin{array}{c|ccccc} & 2N-2 & 2N-1 & 2N & 2N+1 & 2N+2 \\ \hline M_{*,\,1} & \eta & 0 & 0 & 0 & \xi \\ M_{*,\,2} & \lambda & k_1 & k_2 & k_3 & \zeta \end{array} \tag{7.47}$$

With $\eta+\lambda=2$, $\xi+\zeta=2$ and $k_l=k_l^{+1}(c_2^{2m_2})$ for $1\le l\le 3$. Here $\eta+\lambda=2$ follows from $M_{2N-2}=2$ and the other parts follow just as in Subcase 1.

Since $M_{2N-2}=b_{2N-2}=2$ and $M_{2N+2}=b_{2N+2}=2$ hold by the first and last column of (7.47), we have by Lemma 7.5

$$\begin{aligned} M_{2N+2}-M_{2N+1}+\cdots-M_1+M_0 &= b_{2N+2}-b_{2N+1}+\cdots-b_1+b_0, &(7.48)\\ M_{2N-2}-M_{2N-3}+\cdots+M_1-M_0 &= b_{2N-2}-b_{2N-3}+\cdots+b_1-b_0, &(7.49)\end{aligned}$$

Subtracting (7.49) from (7.48) and using Theorem 3.1 together (7.47), we have

$$\begin{aligned} 2-k_3+k_2-k_1 &= M_{2N+2}-M_{2N+1}+M_{2N}-M_{2N-1}\\ &= b_{2N+2}-b_{2N+1}+b_{2N}-b_{2N-1}=4.\end{aligned}$$

This implies

$$k_2-k_1-k_3=2. \tag{7.50}$$

As in (7.43), we have

$$\chi(c_2^{T(c_2)})=k_2-k_1-k_3=2. \tag{7.51}$$

Now by (5.8) and (7.44), we have

$$\hat{\chi}(c_2)=\frac{1}{T(c_2)}\sum_{1\le m\le T(c_2)}\chi(c_2^m)\le\frac{T(c_2)+1}{T(c_2)}. \tag{7.52}$$

We assume

$$\frac{\vartheta_j}{2\pi}=\frac{r_j}{\lambda s_j},\quad r_j,\,\lambda,\,s_j\in\mathbf{N},\ \ (r_j,\,\lambda s_j)=1,\ (s_1,\,s_2)=1,\ j=1,2. \tag{7.53}$$

By (7.2) and Lemma 7.1, we have

$$\hat{i}(c_2)=i(c_2)-2+\frac{\vartheta_1}{\pi}+\frac{\vartheta_2}{\pi}\equiv 2q+\frac{2r_1}{\lambda s_1}+\frac{2r_2}{\lambda s_2}>2, \tag{7.54}$$

where we denote by $2q=i(c_2)-2\in 2\mathbf{N}_0$.

Note that $T(c_2)=\lambda s_1 s_2$. Thus Multiplying both sides of (7.54) by $\frac{T(c_2)}{2}$ yields

$$\frac{T(c_2)\hat{i}(c_2)}{2}=q\lambda s_1s_2+r_1s_2+r_2s_1>\lambda s_1 s_2.$$

Hence

$$\frac{T(c_2)\hat{i}(c_2)}{2}=q\lambda s_1s_2+r_1s_2+r_2s_1\ge\lambda s_1 s_2+1. \tag{7.55}$$

Now by (7.52) and (7.55), we have

$$\frac{\hat{\chi}(c_2)}{\hat{i}(c_2)}\le\frac{T(c_2)+1}{T(c_2)\hat{i}(c_2)}=\frac{\lambda s_1s_2+1}{2(\lambda s_1s_2+1)}=\frac{1}{2}. \tag{7.56}$$

This together with (7.42) yields the contradiction (7.46). Hence the theorem holds in this subcase.

Case 2. *We have* $M_{c_2} = R(\vartheta_1) \diamond N_1(-1, 1)$ *with* $\frac{\vartheta_1}{\pi} \in (0, 2] \cap \mathbf{Q}$.

By Theorem 4.1, we have

$$i(c_2^m) = m(i(c_2) - 1) + 2\mathcal{E}\left(\frac{m\vartheta_1}{2\pi}\right) - 1, \quad \forall m \in \mathbf{N}, \tag{7.57}$$

with $i(c_2) \in 2\mathbf{N}$. Note that

$$i(c_2^m) - i(c_2) \in \begin{cases} 2\mathbf{N}_0 & \text{if } m \in 2\mathbf{N} - 1, \\ 2\mathbf{N}_0 + 1 & \text{if } m \in 2\mathbf{N}, \end{cases} \tag{7.58}$$

One can check that (7.25)-(7.27) still hold in this case.

Note that by the definition of $2m_2$, we have $2m_2\frac{\vartheta_1}{2\pi} \in \mathbf{Z}$. Thus we have

$$\nu(c_2^{2m_2}) = 3, \qquad i(c_2^{2m_2}) = 2N - 1, \tag{7.59}$$

where the latter holds by (7.10)-(7.11) and $i(c_2^{2m_2})$ is odd. Note that by (7.21) and (iii) of Proposition 2.6, we have

$$\overline{C}_{2N-1}(E,\ c_2^{2m_2}) = 0 = \overline{C}_{2N+2}(E,\ c_2^{2m_2}). \tag{7.60}$$

By (7.57) and Lemma 7.1, we have

$$i(c_2^{2m_2+1}) \geq 2N + 2, \qquad i(c_2^{2m_2+m}) \geq 2N + 5, \quad \forall m \geq 2. \tag{7.61}$$

Here the latter holds as in (7.29). Thus by the same argument as in Case 1, we have (7.33).

Now we have the following diagram.

$$\begin{array}{c|cccc} & 2N-1 & 2N & 2N+1 & 2N+2 \\ \hline M_{*,\,1} & 0 & 0 & 0 & \xi \\ M_{*,\,2} & 0 & k_1 & k_2 & \zeta \end{array} \tag{7.62}$$

With $\xi + \zeta = 2$ and $k_l = k_l^{-1}(c_2^{2m_2})$ for $1 \leq l \leq 2$. The first column follows by (7.15), (7.26)-(7.27) and (7.60). The other parts follows just as in (7.38).

Hence we have $M_{2N+2} = b_{2N+2} = 2$ and $M_{2N-1} = b_{2N-1} = 0$. Thus as in Subcase 1.1, we obtain

$$2 - k_2 + k_1 = M_{2N+2} - M_{2N+1} + M_{2N} = b_{2N+2} - b_{2N+1} + b_{2N} = 4. \tag{7.63}$$

This implies

$$k_1 - k_2 = 2. \tag{7.64}$$

Note that by Lemma 5.2 and (7.58), $T(c_2)$ is even. Hence by (7.57), $T(c_2)|2m_2$ and $i(c_2^{2m_2})$ is odd, we have

$$\chi(c_2^{T(c_2)}) = k_1 - k_2 = 2. \tag{7.65}$$

We assume

$$\frac{\vartheta_1}{2\pi} = \frac{r_1}{s_1}, \quad r_1,\, s_1 \in \mathbf{N}, \;\; (r_1,\, s_1) = 1. \tag{7.66}$$

By (7.57) and Lemma 7.1, we have

$$\hat{i}(c_2) = i(c_2) - 1 + \frac{\vartheta_1}{\pi} \equiv q + \frac{2r_1}{s_1} > 2, \tag{7.67}$$

where we denote by $q = i(c_2) - 1 \in 2\mathbf{N}_0 + 1$.

Note that $T(c_2) = \frac{2s_1}{(2,\, s_1)}$. Thus Multiplying both sides of (7.67) by $\frac{T(c_2)}{2}$ yields

$$\frac{T(c_2)\hat{i}(c_2)}{2} = \frac{T(c_2)}{2}\left(q + \frac{2r_1}{s_1}\right) > T(c_2).$$

Since both the second and the third terms are integers, we have

$$\frac{T(c_2)\hat{i}(c_2)}{2} \geq T(c_2) + 1. \tag{7.68}$$

Note that (7.44) still holds. In fact, by Theorem 4.1, we have $\nu(c_2^m) \leq 1$ for $m \neq 0 \bmod s_1$. Hence (7.44) hold for these m by Proposition 2.6. Now if $T(c_2) = s_1$, (7.44) is true. If $T(c_2) = 2s_1$, i.e., s_1 is odd, we only need to consider $\chi(c_2^{s_1})$, then $\nu(c_2^{s_1}) = 2$ and $i(c_2^{s_1})$ is even by (7.58). Thus $\chi(c_2^{s_1}) \leq 1$ holds by Proposition 2.6 and (5.7).

By (7.44), (5.8), (7.65) and (7.68), we have

$$\frac{\hat{\chi}(c_2)}{\hat{i}(c_2)} \leq \frac{T(c_2)+1}{T(c_2)\hat{i}(c_2)} \leq \frac{1}{2}. \tag{7.69}$$

This together with (7.42) yields the contradiction (7.46). Hence the theorem holds in this case.

Case 3. *We have $M_{c_2} = R(\vartheta_1) \diamond N_1(1,\, -1)$ with $\frac{\vartheta_1}{\pi} \in (0,\, 2] \cap \mathbf{Q}$.*

By Theorem 4.1, we have

$$i(c_2^m) = m(i(c_2) - 1) + 2\mathcal{E}\left(\frac{m\vartheta_1}{2\pi}\right) - 1, \quad \forall m \in \mathbf{N}, \tag{7.70}$$

with $i(c_2) \in 2\mathbf{Z}+1$. Hence by (6.1), we have $i(c_2) \geq 3$. By p.340 of [Lon2], we have $2S^+_{P_{c_2}}(1) - \nu(c_2) \geq -1$. Thus by (7.12)-(7.13), we have

$$i(c_2^{2m_2-m}) + \nu(c_2^{2m_2-m}) \;\leq\; 2N - 2, \quad \forall m \in \mathbf{N}, \tag{7.71}$$

$$i(c_2^{2m_2+m}) \;\geq\; 2N + 3, \quad \forall m \in \mathbf{N}, \tag{7.72}$$

Note that the same argument as in Case 2 implies that (7.59)-(7.60) still hold. By (7.32), Theorem 3.2, (7.60), (7.71)-(7.72) and Proposition 2.1, we have (7.33) here. By the same argument as in Case 2, (7.62) holds here with $k_l = k_l^{+1}(c_2^{2m_2})$ for $1 \leq l \leq 2$. Then (7.63)-(7.64) hold.

Note that $T(c_2)|2m_2$ and $i(c_2^{2m_2})$ is odd by (7.70), we have

$$\chi(c_2^{T(c_2)}) = k_1 - k_2 = 2. \tag{7.73}$$

We assume

$$\frac{\vartheta_1}{2\pi} = \frac{r_1}{s_1}, \quad r_1,\, s_1 \in \mathbf{N}, \;\; (r_1,\, s_1) = 1. \tag{7.74}$$

By (7.70) and Lemma 7.1, we have

$$\hat{i}(c_2) = i(c_2) - 1 + \frac{\vartheta_1}{\pi} \equiv q + \frac{2r_1}{s_1} > 2, \tag{7.75}$$

where we denote by $q = i(c_2) - 1 \in 2\mathbf{N}$.

Note that $T(c_2) = s_1$. Thus Multiplying both sides of (7.67) by $\frac{T(c_2)}{2}$ yields

$$\frac{T(c_2)\hat{i}(c_2)}{2} = \frac{T(c_2)}{2}\left(q + \frac{2r_1}{s_1}\right) > T(c_2).$$

Note that $q \in 2\mathbf{N}$, hence both the second and the third terms are integers, hence we have

$$\frac{T(c_2)\hat{i}(c_2)}{2} \geq T(c_2) + 1. \tag{7.76}$$

Note that (7.44) still holds since $\nu(c_2^m) \leq 1$ for $m < T(c_2)$. Thus (7.44), (7.73) and (7.76) imply (7.69) holds. This together with (7.42) yields the contradiction (7.46). Hence the theorem holds in this case.

Case 4. *We have $M_{c_2} = R(\vartheta_1) \diamond P$ with $\frac{\vartheta_1}{\pi} \in (0,\, 2] \cap \mathbf{Q}$ and $P \in Sp(2)$ is hyperbolic.*

In this case, by Theorem 4.1, the index iteration formula coincide with that of Case 2 or Case 3 when $i(c_2) \in 2\mathbf{N}$ or $i(c_2) \in 2\mathbf{N} + 1$ respectively. Thus the same proof of Case 2 and 3 imply our theorem is true in this case.

Case 5. *$M_{c_2} = R(\vartheta_1) \diamond R(\vartheta_2)$ with $\frac{\vartheta_1}{\pi} \in (0,\, 2] \cap \mathbf{Q}$ and $\frac{\vartheta_2}{\pi} \in (0,\, 2] \setminus \mathbf{Q}$.*

In this case, by Theorem 4.1, we have

$$i(c_2^{2m_2}) \in 2\mathbf{N}, \qquad \nu(c_2^{2m_2}) = 2. \tag{7.77}$$

Then by the proof of Lemma 7.6 together with Propositions 2.1 and 2.6, we have

$$1 \geq \dim \overline{C}_{2N}(E, c_2^{2m_2}) = M_{2N} \geq b_{2N} = 2. \tag{7.78}$$

This contradiction proves the theorem in this case.

Case 6. *M_{c_2} dose not contain $R(\vartheta)$ with $\frac{\vartheta}{\pi} \in (0, 2] \cap \mathbf{Q}$ and $N_1(1, 1)$, $N_1(-1, -1)$.*

In this case, by Theorem 4.1, we have

$$\nu(c_2^{2m_2}) \leq 2.$$

Moreover, if the equality holds, M_{c_2} must be one of the following

$$N_1(1, -1)^{\diamond 2}, \quad N_1(1, -1) \diamond N_1(-1, 1), \quad N_1(-1, 1)^{\diamond 2}, \quad N_2(\omega, b),$$

with $N_2(\omega, b) = \begin{pmatrix} R(\vartheta) & b \\ 0 & R(\vartheta) \end{pmatrix}$ and $\frac{\vartheta}{\pi} \in (0, 2] \cap \mathbf{Q}$. Hence by Theorem 4.1, we have

$$i(c_2^{2m_2}) \in 2\mathbf{N}.$$

Then the same argument as in Lemma 7.6 shows the theorem holds in this case.

Combining all the above cases, we obtain Theorem 1.5. ∎

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